Essays in corporate financing and investment under uncertainty

Della Seta, M.

Publication date:
2011

Document Version
Publisher's PDF, also known as Version of record

ESSAYS IN CORPORATE FINANCING AND INVESTMENT UNDER UNCERTAINTY

MARCO DELLA SETA

ESSAYS IN CORPORATE FINANCING AND INVESTMENT UNDER UNCERTAINTY

Proefschrift

ter verkrijging van de graad van doctor aan Tilburg University, op gezag van de rector magnificus, prof. dr. Ph. Eijlander, in het openbaar te verdedigen ten overstaan van een door het college voor promoties aangewezen commissie in de aula van de Universiteit op vrijdag 14 oktober 2011 om 12.15 uur door

Marco Della Seta

door Marco Della Seta, geboren op 24 april 1979 te Spoleto, Italië

Promotor: prof. dr. Peter M. Kort
Copromotor: dr. Sebastian Gryglewicz

Acknowledgements

Whenever I pack before leaving a place, one thing inevitably happens. I forget to take with me something important. Yesterday, when I left Tilburg, was not an exception and now I am staring at my suitcase hopeless to find signs of a rational design. Writing these acknowledgements is like packing. Years of life must fit in a limited space, and I know for sure that I will forget to mention someone or something important.

I cannot forget, however, to thank my supervisors Peter Kort and Sebastian Gryglewicz. They had to guide me through this not always easy journey and to cope with my swinging mood. The latter task was maybe the most difficult. Peter and Sebastian are coauthors of chapters 3 and 4 of this thesis. I am grateful to the members of the committee (in rigorous alphabetical order, Kuno Huisman, Jan Potters, Norman Schürhoff, and Utz Weitzel) for reading this manuscript. They gave me helpful suggestions to further improve my work.

I am also grateful to CentER and the department of Econometrics for providing an excellent research environment and to the entire admin staff. Coming from the place where I am from, I could not even imagine that people working in the administration could be also efficient. Thanks to them I discovered a new world.

Now, it would be time to mention friends, more than friends, and almost friends that shared the daily life with me in Tilburg making it enjoyable. But, as I said at the beginning, it is very likely that I will forget someone important. So, I will skip this part reminding, however, that people which do not appear here were the most important during these years. Let me thank, instead, two persons that, without being here, were always with me. My father, who likes to read about economics but still thinks that the best way to boost a country's economy is to waste public money. And my mother, who finds more and more difficult to say goodbye at the airport whenever I leave. Be strong mum, it will be like this still for some years to come.

Marco Della Seta
Perugia, July 18th 2011

Contents

Chapter 1

Introduction

Economic decisions are rarely now or never. In many real life situations a third way is available to individuals: Waiting. Waiting is often an optimal choice because it gives the opportunity to observe the evolution of the economic environment and to take a more informed decision. The possibility to wait and see is valuable in the presence of uncertainty and when the decision under consideration implies consequences which are, at least to some degree, irreversible. In financial economics, the opportunity to wait associated with the right but not the obligation to undertake an action is at the basis of the notion of "option". This thesis is composed of three essays in which the option approach is used to model different economic problems.

The first essay, "Cash and competition", studies the effects of product market competition of firms' cash holdings. The second essay, "Willingness to wait under risk and ambiguity: Theory and experiment", examines how risk and ambiguity influence the optimal timing of option exercise. The third essay, "Learning investment", analyzes the optimal investment policy in technologies that involve a process of learning by doing. The three essays study substantially different economic problems but are related by a common theme. Agents maximize their value by choosing the timing of an irreversible action in an uncertain environment. As it immediate to understand, such common theme finds a potentially unlimited number of applications, which are not restricted to economic problems.

After all, the entire human existence is characterized by timing decisions which are taken in condition of uncertainty and are to some extent irreversible. In the remainder of this Introduction I will present an overview of the three essays.

The starting point of the first essay, "Cash and Competition", is one of the most interesting facts in recent corporate finance, that is the dramatic increase in cash holdings of US corporations in the last thirty years. The aim of this essay is to study how and by which mechanisms the intensity of product market competition affects firms' cash holdings. The motivation for this study is that, potentially, competition has profound influence on firms' willingness to hold cash. The available empirical evidence shows that the documented increase in cash was mainly driven by changes in the business conditions and firms' characteristics. Competition not only is one a key determinant of the business environment but also, by triggering endogenous selection mechanisms, it can indirectly shape the characteristics of the pool of incumbent firms. Hence, the increase in competition documented in the post WWII period, is likely to have had a major impact on the incentives to hold cash reserves.

The model studies an imperfectly competitive industry with a large number of firms. Firms make entry, exit, and pricing decisions and choose their optimal capital structure to exploit the tax-benefits of debt. The intensity of competition depends on the ability to set a price above the marginal cost of production, as determined by the degree of product substitutability. Firms are subject to idiosyncratic shocks which determine their productivity level and profits. Because of capital market imperfections, access to external finance is restricted and firms that have no means to cover their payments are liquidated even if they are still profitable in a long-run perspective. To prevent this possibility firms hoard cash.

The solution shows that cash holdings positively depend on two components. A cost component, represented by the discounted stream of fixed costs of production and interest payments on debt, and an option component, which captures the option value to remain active in the market in periods of negative profitability. The solution has an intuitive interpretation. Since firms use liquid asset to cover losses in bad times, cash depends on both the stream of fixed cost, the cost component, and on the willingness to cover these costs with internal resources when profitability is low, the option component.

Competition affects the optimal amount of cash via two contrasting channels. First, it increases the option value to remain in the market, and this has a upward effect on cash. Second, it induces the firms to reduce the cost component, and this has a downward effect on cash. The economic intuition is as follows. The effect of competition is to decrease expected profits and to increase volatility. With a higher volatility the option value to remain in the market (the option component) is larger and firms are willing to absorb greater losses prior to declaring default. For this reason, they want to hold more cash. On the other hand, lower expected profits and a higher volatility together increase the risk of default inducing the firms to adopt a more debt-conservative capital structure. Other things being equal, lower debt payments reduce the fixed costs (the cost component) and exert a downward pressure on the optimal amount of cash reserves.

The model generates two main predictions. First, although the overall effect is potentially ambiguous, under realistic conditions cash increases with competition. Second, there is a negative relation between cash and debt. This happens because, when the option value to remain in the market is large, firms have a more compelling need to increase their chances of survival in bad times and increase their cash balance to be able to withstand negative shocks. At the same time, firms adjust their capital structure by reducing fixed interest payments on debt to limit losses in periods of low profitability. By increasing the option value to remain in the market, the effect of competition is to exacerbate the negative relation between cash and leverage. The predictions of the model are largely consistent with the available empirical evidence.

The second essay, "Willingness to wait under risk and ambiguity: Theory and Experiment", studies both theoretically and experimentally the distinct roles of two forms of uncertainty, risk and ambiguity, on the optimal timing of option exercise. While studies on financial and non financial options have mainly considered uncertainty as risk, it is well known that risk is not the only form uncertainty encountered by individuals. In fact, the academic literature distinguished between uncertainty with known probabilities, known as risk, and uncertainty with unknown probabilities, known as ambiguity. Experimental and theoretical studies documented the behavioral significance of this distinction and examined its implications in several economic settings. This second essay is the first attempt to predict and test, in a unifying framework, the effects of risk and ambiguity on the optimal timing of option exercise.

The first step of this work is to develop a new theoretical model of optimal option exercise in which both risk and ambiguity are present. The basic structure of the model is as follows. A decision maker holds the opportunity to invest in a project by paying a fixed cost. The value of the project grows deterministically over time but, at each instant, the option to invest can disappear at an exogenously specified expiration rate. If the decision maker invests before the expiry of the option, he obtains a payoff equal to the current value of the project minus the investment cost, while he gets nothing otherwise. Thus, there is a value in delaying the investment, because the payoff is growing over time, but waiting involves an opportunity cost because the investment option can vanish at instant with positive probability. There are two possible states of the world. In the good state, the expiration rate is low while it is high in the bad state. The true value of the expiration rate is unknown at the initial date but the decision maker can learn about the true state of the world. If time progresses and the investment opportunity does not expire, the decision maker can infer that the state of the world is more likely to be good, and he updates his beliefs accordingly.

This setting allows distinguishing between a risky and an ambiguous scenario. In the risky scenario, the decision maker knows the relative probability of the expiration rate being high or low. Risk is given by the spread between high and low expiration rates, for the expected expiration rate being constant. In the risky scenario, the decision maker has imprecise information about the probability of the two states of the world. He only knows that this probability lies within a certain interval, and ambiguity is measured by the size of probability interval. The model delivers the following predictions. First, risk delays investment. The reason is that, when the spread between high and low expiration rates becomes larger, the non expiration of the option to invest during a given time interval is a more reliable signal that the state of the world is in fact good one. Thus, the upside potential of the option is larger and the decision maker waits for a higher project value before investing. The effect of ambiguity depends on the decision maker's attitude towards ambiguity. If he is ambiguity averse, investment is undertaken sooner. Since investment yields a certain payoff while waiting involves an uncertain prospect, an ambiguity adverse decision maker, who dislikes the uncertainty associated with the waiting region, prefers to invest sooner. In contrast, an ambiguity seeking decision maker is more willing to face uncertainty and waits longer to obtain a larger payoff.

The predictions of the model are tested in a laboratory experiment through three treatments. In the first treatment, called *Benchmark*, subjects know the values of the high and low expiration rates and the relative probability of the two states of the world. In the second treatment, called *Risk*, the spread between the high and low expiration rates (our measure of risk) is increased compared to *Benchmark*. In the third treatment, called *Ambiguity*, the values for the high and low expiration rates are as in *Benchmark* but subjects do not have any information about the relative probability of the two states of the world. Experimental data strongly support the theoretical prediction about risk. In the treatment *Risk*, the investment decision is delayed compared to *Benchmark*. Somewhat surprisingly, also the investment decision in Ambiguity is delayed compared to *Benchmark*. According to the model predictions, this is a sign of ambiguity seeking. The robustness of the latter result is tested in another treatment, called *Mild Ambiguity*, in which growth and expiration rates as in *Benchmark* and *Ambiguity* but subjects have a partial information about the relative probability of the states of the world. Data reveal that in *Mild Ambiguity* investment is still delayed compared to *Benchmark*, though the effect is substantially weaker than in *Ambiguity*. Overall, we find a weak confirmation of an ambiguity seeking attitude.

The third and last essay, "Learning Investment", studies investment in technologies that involve a process of learning by doing. Specifically, it investigates the optimal timing and scale of investment when demand is uncertain and marginal costs decrease with cumulative production. The literature on investment under uncertainty mainly focuses on the optimal timing of investment. This essay also investigates the choice of optimal capacity. The motivation for studying the joint determination of timing and scale is the existence of a trade-off. When the scale of investment is flexible but the timing is not, the presence of the learning curve implies that firms should invest in a larger capacity. On the other hand, when the timing is flexible but the scale is fixed, the learning curve accelerates investment. These two observations suggest that investment should occur early and on a large scale to maximize the benefit of learning. However, investing early, that is, investing at the moment that levels of demand or productivity are still low implies that only small scale projects are feasible. At the same time, a large scale investment typically requires higher demand or productivity and entails a longer waiting time, so that some profits are foregone. Therefore, an optimal investment strategy requires finding a balance between timing and scale that allows firms to benefit from the learning curve but, at the same time, it is not too costly in the short run.

The resolution of the timing-scale trade-off depends on the steepness of the learning curve. Under slow learning investment occurs relatively late and on a larger scale, whereas under fast learning it occurs early and on a smaller scale. In the latter case firms do not need large production rates to substantially reduce marginal costs. Hence, it is optimal to invest soon and install a small capacity. The opposite holds under slow learning, because then optimality implies that a firm should install a larger capacity to reduce marginal costs sufficiently within a given amount of time. Given the larger project size, investment is delayed. It turns out that, where timing is accelerated, scale is inversely U-shaped in the steepness of the learning curve.

To take advantage of learning benefits, firms may undertake learning investments even when current revenue rates are below costs. Thus, the optimal investment rule implies that the firm will incur losses at early stages of production. The analysis indicates that, typically, the present value of expected initial losses is large and is the largest for moderate learning rates. For steep learning curves, the initial level of losses is similar but, because of rapid learning, the break-even point is reached sooner. Third, the losses incurred in early production stages can easily dwarf the initial investment outlays to set up the production facility. Overall, these findings indicate that learning investments can be financially very demanding for firms. This is especially true for technologies with intermediate learning curves.

Learning investment may be particularly exposed to downside risk. New technologies may be superseded by newer technologies, turn out unmarketable, or flawed. To analyze how downside risk affects optimal investment, we extend the model by introducing the possibility that with positive probability an event occurs that results in the death of the project. We show that learning investment is very sensitive to this type of risk. Investment is significantly delayed and scale increases with the occurrence of even small levels of downside risk. In contrast, timing and scale of non-learning investment are very insensitive to this type of risk. Furthermore, the value of investment projects with learning effects is decreased more by downside risk. Interestingly, the effects on learning investment are strong for moderate learning curves and steeper curves do not amplify them further. The explanation is related to the initial losses associated with learning investment, which are similar for these cases. Furthermore, the threat that the project expires before any profits materialize, distorts learning investment and the long-term benefits of learning cannot be fully exploited.

Chapter 2

Cash and Competition

2.1 Introduction

In two distinct empirical studies, Opler et al. (1999) and Bates et al. (2009) report that U.S. corporations hold substantial amounts of cash reserves. Bates et al. (2009) also document a dramatic increase in the cash holdings of the typical firm in the period from 1980 to 2006. Despite the growing attention from the academic literature, explaining why firms hold so much cash when there are other options to manage liquidity remains a challenge for the theory of corporate finance. This work does not directly take on this challenge but, starting from the empirical evidence that firms do hold cash, it studies how and by which mechanisms the intensity of product market competition affects firms' cash reserves.

Among the potential determinants of firms' cash holdings, competition is a natural candidate to look at. The empirical analysis of Bates et al. (2009) reveals that the documented increase in cash was mainly driven by changes in the business conditions and firms' characteristics. Competition not only is *per se* a key aspect of the business environment but, through endogenous selection mechanisms, it may also indirectly shape the characteristics of the pool of surviving firms. Furthermore, several indicators consistently suggest that the intensity of competition has steadily increased in the last forty years (for example, Comin and Philippon (2005) and Irvine and Pontiff (2009), among others). This fact is likely to have had a major impact on the incentives to hold cash reserves.

I study an industry with a large number of competitors, in which firms are subject to individual productivity shocks and hold an option to default whenever market conditions become unfavorable. Firms make entry, exit, and pricing decisions and choose their optimal capital structure to exploit the tax-benefits of debt. The intensity of competition depends on the ability to set a price above the marginal cost of production, as determined by the degree of product substitutability. Capital markets are imperfect and access to external finance is restricted. Firms that have no means to cover their payments are liquidated even if profitable in a long-run perspective. To prevent this possibility, firms accumulate cash. Covering losses to remain alive may not be the only reason why firms hold cash. The presence of cash holdings within the firm can also be explained by the need to finance profitable investment opportunities when access to external finance is restricted, or by agency conflicts between managers and shareholders (Jensen (1986)). The modeling choice of this work is based on the evidence that firms mainly use cash to withstand liquidity shortfalls in bad times (Opler et al. (1999), Bates et al. (2009) and, in particular, Lins et al. (2010)), while both investment and agency motives seem to be of poor empirical relevance (Opler et al. (1999), Bates et al. (2009) and Lins et al. (2010)).

The properties of the model are investigated in the stationary industry equilibrium. In equilibrium, there is a time-invariant distribution function which describes the productivity of incumbent firms, and aggregate variables are endogenously determined and constant over time. In this setting, I study the long-run effects of exogenous changes in the intensity of product market competition on the optimal amount of cash reserves. The equilibrium approach has two important advantages. First, it is consistent with the idea that variations in the intensity of competition often depend on exogenous shocks (as the removal of regulatory barriers, the reduction of legal and administrative restrictions on entry, or the opening of new markets) which require a certain time for the firms to adjust. Second, it captures the fact that competitive pressure can affect cash holdings not only via the direct effects on the business environment but also through changes in firms' characteristics induced by endogenous self-selection mechanisms.

After solving for the industry equilibrium, I provide the expression for the optimal amount of cash in closed-form. The solution shows that cash holdings positively depend on two components. A cost component which is given by the discounted stream of fixed costs of production and interest payments on debt, and an option component, which captures the value of remaining active in the market in periods of negative profitability. I show that the effect of competition is to increase profit volatility and to reduce the expected profit for the average firm. This gives rise to two contrasting effects on cash holdings. On the one hand, higher volatility triggers the standard real options effect and increases the value of the option component. Since the exit decision is irreversible and currently adverse market conditions can rapidly turn positive, the option value to remain active in the market becomes more valuable. For this reason, firms are willing to absorb larger losses prior to declaring default and need greater amounts of cash reserves. On the other hand, lower expected profits and higher volatility together reduce the net benefits of debt inducing the firms to adopt a more debt-conservative capital structure. Other things being equal, lower debt payments exert a downward pressure on the cost component and tend to reduce the optimal amount of cash reserves.

Hence, competition affects the optimal amount of cash via two opposing forces. The first is the increase in the option value to remain in the market, which has an upward effect on cash. The second works through the reduction in the debt payments and has a downward effect. Although the overall effect is potentially ambiguous, I show that, under realistic conditions, cash increases with competition. In the main text, I provide some technical explanations to motivate this finding. Here, I restrict my attention to a more general argument. In the model, firms can freely choose their capital structure before the entry date. This means that they have a high degree of flexibility to optimally adjust their debt payments to the expected market conditions. In reality, constrained firms may not have such flexibility. When capital markets are imperfect, a sensible reduction in debt payments as a response to a riskier economic environment can be difficult to achieve either because access to equity financing is more costly than debt or because, due to the constraint itself, the original level of debt is already low. A lack of financial flexibility impairs the functioning of the cost component channel and makes the upward effect due to the option component more likely to prevail. If this is the case, cash reserves are expected to increase with the intensity of competition.

The model also predicts a negative relation between cash and debt. This effect is driven by the option component. When the value to remain active in the market becomes larger, firms increase their cash balance to be able to withstand negative shocks. At the same time, they adjust their capital structure by reducing leverage. A more debt-conservative capital structure lowers fixed interest payments, reduces the risk of liquidation and increases the probability of survival in bad times. By increasing the option value and reducing the net benefits of debt, the effect of competition is to exacerbate this negative relation.

The identified relation between competition, idiosyncratic volatility, capital structure and cash holdings is consistent with a number of empirical facts documented in the literature. Over the time horizon investigated by Bates et al. (2009), idiosyncratic volatility displayed a substantial increase and was the major source of firm-level dynamics (Campbell et al (2001), Chaney et al. (2005), Comin and Philippon (2005)). Irvine and Pontiff (2009) prove that the increase in volatility is at least partly attributable to a more intense product market competition, providing empirical ground for the main channel indentified in this work. At the same time, corporate cash holdings increased steadily and, as reported in Bates et al. (2009), leverage for the median firm decreased sensibly over the years.[1] Consistently, Opler et al. (1999) and Bates et al. (2009) find that there exists a negative relation between cash holdings and leverage. The model provides a theoretical foundation for bringing together these pieces of evidence. When idiosyncratic shocks are the main source of uncertainty, a more intense product market competition raises firm-level volatility, increases the option value to remain in the market and reinforces the precautionary motive for holding cash. Also, the option value to remain in the market can generate a negative relation between cash and leverage.

Firms' cash policy received increasing attention from the academic lit-

[1] However, in the sample of Bates et al. (2009), the reduction in leverage for the average firm is less pronounced.

erature (for example, Opler et al. (1999), Bates et al. (2009), Almeida et al. (2004), Acharya et al. (2007) and Lins et al. (2010)). The closest work to mine is a recent paper by Morellec and Nikolov (2011) which also examines the effects of product market competition on firms' cash holdings. Their theoretical predictions suggest, and the empirical analysis confirms that the trend in cash holdings documented by Bates et al. (2006) is at least partly explained by a competition effect. The focus of Morellec and Nikolov (2011), however, is mainly directed to the empirical analysis. This work complements their study by identifying the mechanisms by which competition influence cash holdings. Furthermore, it identifies in the option value to remain active in the market the key to explain the negative relation between cash and leverage observed in the data. Another related work is the paper by Murto and Terviö (2011), which introduces a liquidity constraint in a dynamic exit model and characterizes the optimal default and dividend policy. Murto and Terviö (2011) examine the steady state equilibrium of a competitive industry and show that the liquidity constraint not only has the direct effect of imposing inefficient exit but also creates a price distortion that leads to inefficient survival. Gryglewicz (2011) studies a model with long-term uncertainty and short-term liquidity shocks in which the firm simultaneously chooses cash holdings, capital structure, dividends, and optimal default. These interactions result in a dynamic cash policy in which the firm smoothes dividend payments while cash reserves increase in profitability and are positively correlated with cash flows. Boyle and Guthrie (2003) also introduce credit constraints in a real options model but they investigate a firm's entry choice, in which uncertainty does not affect the ex-post investment cash-flows but only the pre-entry availability of funds to cover investment costs. In an empirical investigation, Frésard (2010) reverses the causal link of this work and studies the effect of cash reserves on market outcomes and firms' performance. He shows that, when competition becomes more intense, cash-rich firms gain market shares at the expense of industry rivals.

To derive the industry equilibrium, I adapt the concept of stationary equilibrium introduced by Hopenhayn (1992) to a dynamic stochastic version of the Dixit-Stiglitz model of monopolistic competition. While Hopenhayn (1992) employs a discrete time framework, my model is in continuous time and is, from a methodological point of view, closer Miao (2005). Zhdanov (2007) develops a continuous time equilibrium model to study the relation between competition and the optimal investment and financing strategies. In his analysis, however, firms are subject to industry-wide uncertainty so that the resulting equilibrium is non-stationary. Novy-Marx (2007) also investigates a competitive model in continuous time with a non-stationary equilibrium. Industry models in discrete time with non-stationary equilibria include Ericson and Pakes (1995) and Abbring and Campbell (2010).

I organize this work as follows. Section 2.2 presents the general structure of the model and describes the product and capital markets. Section 2.3 solves model while taking the capital structure as exogenous. Section 2.4 investigates the optimal capital structure model. Finally, Section 2.5 concludes. Proofs are in Appendix.

2.2 The model

2.2.1 Production and demand

Time is continuous and indexed by $t \in [0, \infty)$. At each instant a representative consumer maximizes a utility function over a *continuum* of goods indexed by υ:

$$U = \left[\int_{\upsilon \in \Upsilon} q^{\theta} (\upsilon) \, d\upsilon \right]^{\frac{1}{\theta}}. \tag{2.1}$$

Utility is maximized subject to the budget constraint:

$$\int_{\upsilon \in \Upsilon} p (\upsilon) \, q (\upsilon) \, d\upsilon \leq Y, \tag{2.2}$$

where $q(\upsilon)$ is the consumption of good of quality υ, Υ is the set of varieties produced in the industry, $\theta \in (0, 1)$ is the degree of product substitutability and Y is the exogenous expenditure, normalized to one hereafter. The intensity of product market competition is parsimoniously captured in the model by the elasticity of substitution between products, which is constant and equal to $\eta = 1/(1 - \theta) > 1$. The focus of this work is to investigate how and by which channels an increase in competition, i.e. a rise in the elasticity of substitution η, affects firms' willingness to hold liquid assets.

As Dixit and Stiglitz (1977) show, the optimal consumption decision for a single good implies that

$$q(\upsilon) = \frac{1}{P}\left(\frac{p(\upsilon)}{P}\right)^{-\eta}, \tag{2.3}$$

where

$$P = \left[\int_{\upsilon\in\Upsilon} p(\upsilon)^{1-\eta}\, d\upsilon\right]^{\frac{1}{1-\eta}} \tag{2.4}$$

is an aggregate price index.

2.2.2 Firms

The production side is characterized by a *continuum* of infinitesimal firms. Each firm produces a single variety using labor as the only input for production. Labor is inelastically supplied and is demanded in quantity

$$l = F + \frac{q}{\psi}, \tag{2.5}$$

where $F \geq 0$ is a fixed component of labor demand common to all firms, and ψ is the firm-specific productivity level. As in Melitz (2003), higher productivity is modeled as producing a symmetric variety at lower marginal costs.

Firms are subject to idiosyncratic shocks to their productivity. This is captured by the fact that ψ follows a geometric Brownian motion:

$$\frac{d\psi_t}{\psi_t} = \mu dt + \sigma dW_t, \tag{2.6}$$

where μ and σ are the proportional drift and volatility. Brownian shocks are assumed to be independent across firms.

Firms set prices to maximize their own profits. Profit maximization yields the optimal pricing rule:

$$p(\psi) = \frac{w}{\theta\psi}, \tag{2.7}$$

where w is the common wage rate also normalized to one, hereafter. I assume that prices can be adjusted at no costs so that (2.7) holds at every instant. It follows that firms generate earnings before tax and interest payments ($EBIT$) equal to:

$$EBIT = \frac{1}{\eta} (P\theta\psi)^{\eta-1} - F. \tag{2.8}$$

Future earnings are discounted at a constant rate ρ.

Beside individual productivity shocks, firms are subject to another source of idiosyncratic uncertainty. At every instant, firms can exit for exogenous reasons not related to their profitability. This event is modeled as a Poisson shock with mean arrival rate λ. Poisson shocks are also assumed to be independent across firms. The possibility of exogenous exit captures in a stylized way the fact each year a number of firms abandon their operations even if they are still profitable (for example, Dunne et al. (1988)). Furthermore, it is necessary to guarantee the existence of a stationary equilibrium. Since the process for the productivity shock is non-stationary, without exogenous death the number of firms with a high productivity could grow unbounded (see also Miao (2005)).

2.2.3 Debt and default

Corporate profits are taxed at a constant rate $\tau \in (0, 1)$ with full loss-offset provisions. Since interest payments are tax-deductible, debt creates tax benefits and firms choose the debt-equity mix that maximizes their value. Indicate by E the value of equity and DBT the value of debt. The total value of the firm, denoted by V, is given by the sum of equity and debt, $V = E + DBT$. Debt has infinite maturity and pays a constant coupon b. Firms can only be net borrowers, which implies that $b \geq 0$. It follows that the instantaneous profit net of taxes and interest payments equals:

$$\pi = (1 - \tau) (EBIT - b). \tag{2.9}$$

Firms have the option to default and exit the industry. Exit is irreversible. As I will show below, the optimal default policy is formulated as a trigger strategy which prescribes that the firm should default on its obligations whenever its productivity ψ falls below an endogenously determined threshold. Define ψ_e as the productivity level such that, if $\psi \geq \psi_e$, it is optimal to remain active in the market while, if $\psi < \psi_e$, it is optimal to default.

In case of default, the firm is liquidated and debt-holders have absolute priority on the productive assets. The liquidation value of the assets is a fraction $(1-\varepsilon)$ of the value of an unlevered and unconstrained firm, where $\varepsilon \in (0,1)$ is the proportional liquidation cost. The value of an unlevered and unconstrained firm, indicated by V_u, equals the discounted stream of profits plus the abandonment option and can be written as:

$$V_u(R) = \sup_{t_u \in \mathcal{T}} \mathbb{E}\left\{\int_0^{t_u} (1-\tau)\, e^{-(\rho+\lambda)t} EBIT dt\right\}, \tag{2.10}$$

where t_u is the optimal abandonment time and the maximization is over the set of possible abandonment times $\mathcal{T}$. The value of equity of a levered firm is given by the discounted stream of profits until the optimally chosen abandonment time t_e,

$$E(R) = (1-\tau) \sup_{t_e \in \mathcal{T}} \mathbb{E}\left\{\int_0^{t_e} e^{-(\rho+\lambda)t} \pi dt\right\}, \tag{2.11}$$

while the value of debt is the stream of coupon payments until default plus the present value at the abandonment time of the unconstrained and unlevered firm,

$$DBT(R) = \mathbb{E}\left[\int_0^{t_e} e^{-(\rho+\lambda)t} b dt\right] + (1-\varepsilon)\, V_u(R) \mathbb{E}\left[e^{-(\rho+\lambda)t_e}\right]. \tag{2.12}$$

The abandonment time t_e is chosen to maximize shareholders' value.

2.2.4 Aggregation

Call N the number of firms currently active in the market and $f(\psi)$ the distribution of the productivity levels of those firms. Since the productivity shock follows (2.6) and firms voluntarily exit when ψ falls below ψ_e, the distribution $f(\psi)$ is defined over the interval $[\psi_e, \infty)$. Using the definition of P and the pricing rule (2.7), the aggregate price index becomes

$$P = \frac{N^{\frac{1}{1-\eta}}}{\theta\ \psi_A}, \tag{2.13}$$

where

$$\psi_A = \left[\int_{\psi_e}^{\infty} \psi^{\eta-1} f(\psi)\, d\psi \right]^{\frac{1}{\eta-1}} \tag{2.14}$$

is the weighted average productivity of the incumbent firms. Substituting (2.13) in (2.8), and using (2.9), yields the following expression for the per-period profit:

$$\pi = (1-\tau) \left[\frac{1}{\eta N} \left(\frac{\psi}{\psi_A} \right)^{\eta-1} - F - b \right]. \tag{2.15}$$

Notice that π depends on the relative strength of the firm in the market, given by the ratio between the idiosyncratic productivity ψ and the industry average productivity ψ_A.

The model is investigated in the long-run stationary equilibrium in which the industry-wide variables N and ψ_A (and therefore P) are constant over time. As in Miao (2005), a law of large numbers for continuous random variables is assumed to hold. This implies that idiosyncratic shocks cancel-out in the aggregate and ensures that the distribution $f(\psi)$ is time invariant. Furthermore, in equilibrium the outflow of firms is offset by the inflow of new competitors, so that the number of incumbents remains constant.

For future reference, I define

$$R = \frac{1}{\eta N} \left(\frac{\psi}{\psi_A} \right)^{\eta-1} \tag{2.16}$$

as the revenue net of taxes and variable cost. In the remainder, the optimal default policy will be defined in terms of a default threshold R_e such that, if $R \geq R_e$, it is optimal for the firm to remain active in the market and to default otherwise. It is easy to show that revenue R follows a geometric Brownian motion:

$$\frac{dR_t}{R_t} = \widetilde{\mu}(\eta)\, dt + \widetilde{\sigma}(\eta)\, dW_t, \tag{2.17}$$

where

$$\widetilde{\mu}(\eta) = (\eta-1)\,\mu + \frac{1}{2}(\eta-2)(\eta-1)\,\sigma^2 \text{ and } \widetilde{\sigma}(\eta) = (\eta-1)\,\sigma. \tag{2.18}$$

The intensity of product market competition affects both the revenue growth rate and volatility. Specifically, the volatility coefficient $\widetilde{\sigma}(\eta)$ increases with the elasticity of substitution η, while the effect on the growth rate $\widetilde{\mu}(\eta)$ is ambiguous. Consider, first, the effect on $\widetilde{\sigma}(\eta)$ and assume that the firm is hit by a positive shock, that is an increase in ψ. According to the pricing rule (2.7), a higher productivity implies a lower optimal price, an improvement in the competitive position of the firm and, therefore, an increase in profits (see equation (2.15)). The magnitude of this effect depends on the elasticity of substitution. When the elasticity of substitution is high, the decrease in price will attract more customers and cause a greater increase in demand and profits. A symmetric reasoning holds for negative shocks. In that case, the increase in price and the decrease in demand and profits will be greater the higher is the elasticity of substitution. Thus, volatility increases with competition.[2]

In contrast, the effect on the growth rate is ambiguous. The reason is that firm's revenue is, in general, a non linear function of ψ. If the revenue function is concave, the growth rate is less than the expected change in productivity. A rise in the elasticity of substitution may increase the concavity of the revenue function and decrease the growth rate. This happens when $\mu < (3/2 - \eta)\,\sigma^2$. When $\mu > (3/2 - \eta)\,\sigma^2$, larger elasticity of substitution either reduces concavity or it increases convexity and, therefore, it increases the growth rate.

2.2.5 Capital market

In a frictionless world, there is no need to hold cash reserves. If solvent in a long-run perspective, a firm will always be able to raise liquidity by issuing either new equity or debt at no costs. In contrast, if access to the capital market is subject to frictions, external funding may not be freely available. For this reason, it can be optimal for the firm to hold a certain amount of cash reserves. To introduce the need for liquidity, I assume that firms

[2] A similar relation between competition and volatility is found in Raith (2003) and Irvine and Pontiff (2009). Boone et al. (2007) and Boone (2009) construct empirical measures of competition based on the idea that, when competition becomes tighter, market shares and profits reallocate faster to the more efficient firms. An analogous mechanism is at work here. Consider two firms with productivity ψ_1 and ψ_2, and assume that $\psi_1 > \psi_2$. The relative difference in profits between the two firms increases with the elasticity of substitution.

can raise external finance only at the initial time $t = 0$.[3] This captures in a stylized way the idea that firms can find it difficult to access the capital market at reasonable conditions (for example, because of problems of asymmetric information or moral hazard) and need, at least to some extent, to rely on internal resources.[4]

Without access to external finance, if a firm incur losses and has no internal resources to meet its payments, it will be liquidated even if current revenue is above the first-best exit threshold, i.e. if $R > R_e$. This is clearly inefficient. To avoid inefficient liquidation, firms hold reserves of liquidity (cash). In practice, cash is not the only mean by which firms can manage idiosyncratic uninsurable shocks. For example, firms could meet their liquidity needs by drawing down bank credit lines. However, as documented by Lins et al. (2010), cash and credit lines are employed to hedge against different risks. While, cash holdings serve as a buffer against cash shortfalls in bad times, credit lines are mainly employed to exploit profitable investment opportunities. Consistent with this evidence, I abstract from credit lines and assume that cash holdings are the only mean to fund operating losses. In the remainder, cash will be indicated by M.

Within the firm, cash reserves earn an interest equal to r. If r is below the discount rate, $r < \rho$, holding cash entails a liquidity premium and is costly for the equity-holders. Then, firms trade-off the costs of holding cash with the benefits stemming from the insurance provided against inefficient liquidation. Here, I follow Mello and Parsons (2000) and Gryglewicz (2011), and assume that cash reserves earn an interest equal to the discount rate, i.e. $r = \rho$. This means that there are no costs of holding cash it is never strictly optimal for the firm to pay out dividends. However, there is a finite amount of cash reserves, indicated by $\overline{M}$, which allows the firm to avoid inefficient liquidation. Firms find it strictly optimal to re-

[3] The assumption that debt can be issued only at the entry date is customary in dynamic contingent claim models of optimal capital structure (for example, Leland (1994), Leland and Toft (1996), Sundaresan and Wang (2007) among others). Here, as in Gryglewicz (2011), I impose the additional restriction that also equity financing is not available after the entry date.

[4] I could also consider a milder form of capital market imperfection. The main intuition of the model would not change.

tain earnings they are at risk of inefficient liquidation (i.e. $M < \overline{M}$), while they are indifferent between retaining and paying out the excess liquidity if $M \geq \overline{M}$.[5] As discussed in Appendix 2.A.5, $\overline{M}$ is the amount of cash reserves whose interest income is just sufficient to cover the worst-case losses under the first-best default policy. To avoid indeterminate scenarios, I assume that, if $M > \overline{M}$, the excess liquidity is paid out to the equity-holders in the form of dividends. Therefore, in the remainder I refer to $\overline{M}$ as the optimal amount of cash.

In reality, holding cash within the firm can be costly, for example, because of agency problem as in Jensen and Meckling (1976). Abstracting from agency considerations the cost of holding cash may arise because of the disadvantage imposed by the double-taxation on internal funds or for the fact that interest corporate cash is taxed at the corporate tax rate, which in general exceeds the personal tax rate on interest income (Faulkender and Wang 2006). At the same time, however, if external investors are not as good as the firm at identifying profitable investment opportunities, holding cash within the firm is a value maximizing strategy. Here I assume that, net of the liquidity risk imposed by the financial constraint, benefits and costs of carrying cash offset each other. This assumption comes at a cost of an upward bias on the predicted optimal amount of firms' cash holdings (firms accumulate so much cash to be perfectly insured against inefficient liquidation) but allows a clearer identifications the mechanisms by which competition affects cash. Since firms are *de facto* unconstrained the valuation problem can be solved by ordinary differential equations which can be solved analytically with standard methods. Whenever cash reserves are not sufficient to surely avoid inefficient liquidation, the value of the securities, namely equity and debt, will also depend on the level of cash reserves and must be found as a (numerical) solution of a *partial* differential equation (see Murto and Terviö (2011)). The assumption of no liquidity premium is further discussed in Section 2.2.7. Since $r = \rho$ holds throughout, ρ is substituted by r, hereafter.

[5] On this point, see also Murto and Terviö (2011).

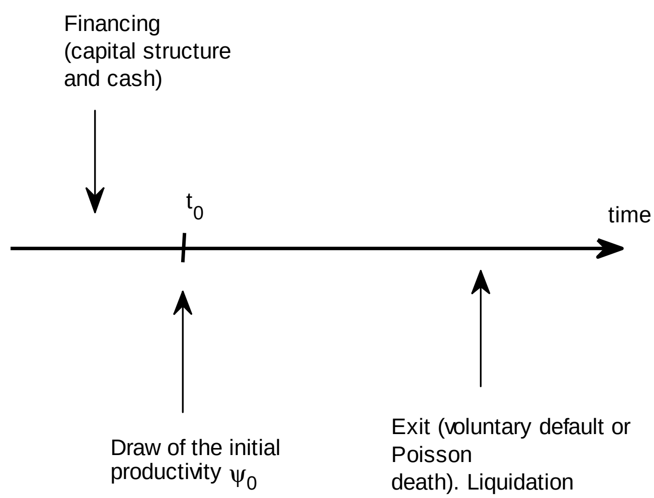

Figure 2.1: Sequence of events and timing decisions.

2.2.6 Entry

At every instant there is an arbitrarily large number of potential producers ready to enter the market. The industry entry rate is indicated by n. Potential entrants freely decide to become active by paying a sunk investment cost I. At the time of entry t_0, firms draw their initial productivity ψ_0 from a uniform distribution defined over a common support $[\underline{\psi}, \overline{\psi}]$.[6] Since firms draw their initial productivity from the same distribution, they are identical ex-ante but differentiate ex-post depending on the evolution of the idiosyncratic shocks. Before knowing the value of the initial productivity, internal equity-holders choose the initial level of liquid assets, indicated by M_0, and the debt-equity mix to maximize their expected value at the entry date. A summary of the timing decisions is found in Figure 2.1.

Consider the initial financing problem and assume that raising external

[6] A uniform distribution is useful to derive closed-form solutions for the stationary equilibrium (see also Miao (2005)).

funds involves a fixed issuance cost equal to $L \geq 0$. Internal equity-holders need to raise external funds to cover the sunk investment cost I, the issuance cost L, and the entry cash reserves M_0. Indicate by E_{-1} and DBT_{-1} the equity and debt value at the entry time, where the subscript "-1" means that I am considering the value before the draw of the initial productivity. If $\omega \in (0,1)$ is the fraction of equity obtained by the external equity-holders, then the following funding condition must hold:

$$I + M_0 = \omega E_{-1} + DBT_{-1} - L. \tag{2.19}$$

Rearranging the above equality, the expected value for the internal equity-holders is found as:

$$(1-\omega)\, E_{-1} = V_{-1} - L - I - M_0, \tag{2.20}$$

where $V_{-1} = E_{-1} + DBT_{-1}$ is the total expected value of the firm.

In a competitive equilibrium, firms enter the market as long as the value of the internal equity-holders is weakly positive, $(1-\omega)\, E_{-1} \geq 0$. Using equation (2.20), this implies that, in a stationary equilibrium, the following entry condition must hold:

$$V_{-1} = L + I + M_0. \tag{2.21}$$

Although there are no costs of holding cash (there is no liquidity premium), equation (2.21) reveals that raising cash is costly for the internal equity-holders because it increases the total cost of entry. To optimally finance the initial investment, credit constrained equity holders need to furnish additional resources compared to the unconstrained case. Indeed, they not only need to finance the initial investment outlay $(L + I)$ but have also to provide the firm with an initial stock of cash reserves (M_0). This is the case because entering the market without internal resources is clearly suboptimal. If a negative shock strikes after the entry date, the firm would be immediately liquidated before any profit materializes. The cost of raising cash it will be called "liquidity cost", hereafter. A consequence of the liquidity cost (not shown in the analysis) is to reduce the number of firms in equilibrium compared to the unconstrained scenario. A higher total cost of entry implies that, for a given expected initial productivity, a lower number of firms can afford to become active in the market.

Furthermore, as it will be shown in Section 2.4, the liquidity cost has the important implication to force constrained firms to issue a suboptimal level of debt.

To find the initial amount of cash, it is useful to recall that, if a firm follows the optimal cash policy, the value of a marginal unit of cash within the firm is larger than or equal to one. To see this, let the value of the firm be explicitly dependent on cash, $V = V(M)$, while other variables are omitted for notational convenience, and consider a firm with cash reserves equal to M. If this firm follows the optimal cash policy, its value must be greater than or equal to the value of a firm which holds cash reserves equal to $M - dM$ and pays a dividend equal to dM, that is $V(M) \geq V(M - dM) + dM$. Rearranging the inequality and letting dM go to zero yields $V'(M) \geq 1$. In absence of liquidity premium, this implies that the marginal value of cash is equal to its face value, $V'(M) = 1$, whenever cash reserves are sufficient to avoid inefficient liquidation, i.e. when $M \geq \overline{M}$.

Consider, now, the choice of M_0. Internal equity-holders choose M_0 to maximize their value. Differentiating (2.20) with respect to M yields $V'(M) = 1$ which it is true for any M larger than or equal to $\overline{M}$.[7] Since raising cash increases the cost of entry, firms will rationally choose the minimum amount of cash that avoids inefficient liquidation, i.e. $M_0 = \overline{M}$. Thus, the entry condition becomes:

$$V_{-1} = L + I + \overline{M}. \tag{2.22}$$

2.2.7 Discussion

Before proceeding with the analysis, it is useful to briefly summarize and discuss the structure of the model. There is an industry with a *contin-*

[7] Condition (2.21) also implies that at the optimum $E'_{-1}(M) = 0$. When a firm is at liquidity risk (i.e. when $M < \overline{M}$), an additional unit of cash brings the firm further away from inefficient liquidation and increases the value equity (net of cash), i.e. $E'_{-1}(M) > 0$. In absence of liquidity premium, it is optimal for the equity holders to retain cash until to the point where the firm becomes *de facto* unconstrained and value of equity equals the discounted stream of profits plus the default option. This occurs for $M = \overline{M}$. Beyond that point additional cash does not further increase the value of equity, $E'_{-1}(M) = 0$ (see also Mello and Parsons (2000)).

uum of firms and an arbitrarily large number of potential entrants. Firms make entry, exit and pricing decisions, and choose their optimal capital structure. Access to credit is restricted and firms hold cash to avoid inefficient liquidation. The model is studied in the long-run stationary equilibrium in which industry-wide variables remain constant. In equilibrium the industry appears as "static" from an aggregate perspective but, at the firm-level, a rich dynamic is still present. Individual firms undergo idiosyncratic productivity shocks and experience changes in their profitability. Some of them optimally decide to exit, some others die due to the Poisson shocks, while new producers enter the industry until the equilibrium entry condition (2.22) is restored. The fact that industry-wide variables are time invariant has two implications. First, incumbent firms do not have any incentive to engage in predatory pricing strategies to force their competitors out of the market.[8] Hence, at each instant, (2.7) is indeed the optimal pricing rule. Second, profitability of the individual firm is determined by the fluctuations of its own productivity and not by the actions of its competitors. This implies that each firm chooses financing and exit policy free from strategic considerations, and its value can be found with the standard methods used for the valuation of a single monopolistic firm.

There are no costs of carrying cash, and firms hold the minimum amount of cash reserves which allows avoiding inefficient liquidation, $\overline{M}$. As shown in Appendix 2.A.5, $\overline{M}$ yields an instantaneous interest income just sufficient to cover the worst-case losses under the first best default policy. This, coupled with the fact that excess liquidity is paid out by assumption, implies that cash reserves will be constant over time and equal to $\overline{M}$. Thus, the optimal cash policy implied by the model is stylized in two respects. First, since there is no liquidity premium, $\overline{M}$ is very large. Even if the predicted optimal cash level is most likely overstated, this captures in a simple fashion the evidence that firms hold so large amounts of cash that, in recent years, net debt (debt minus reserves of liquidity) became negative for the typical firm (see Bates et al. (2009)). The second simplification is that cash reserves remain constant at $\overline{M}$ while in reality they fluctuate with variations in the business conditions. This concern is

[8] This also depends on the fact that firms are infinitesimal and the choice of a single firm has no impact on the aggregate price index.

mitigated by the fact that, as shown by Opler et al. (1999), firms have a target cash level. Since the model studies an industry in its steady-state, $\overline{M}$ can be interpreted as the amount of cash reserves that firms are willing to hold in equilibrium. Finally, the purpose of this work is neither to explain cash holdings dynamics nor to give quantitative predictions about the optimal amount of liquid assets. Rather, the ultimate goal is to study the qualitative effects of competition on cash holdings. In my framework, closed-form solutions for the industry equilibrium and the optimal level of cash can be derived. This allows a clear identification of the channels by which competition affects firms' cash holdings.

2.3 Exogenous leverage

In this section, I solve the model by taking the coupon payment b as exogenously given. In Section 2.4, I will endogeneize the capital structure decision. The motivation for this intermediate step is twofold. First, with an exogenous coupon, the optimal cash policy has a full analytical solution. Second, the predictions of this section hold for firms with no (or lack of) flexibility in choosing their capital structure. Also, leaving aside the debt valuation part, the analysis applies for unlevered firms which produce with fixed costs equal to $F + b$.

2.3.1 Equilibrium

To begin with, I define the long-run stationary equilibrium and prove its existence and uniqueness. This is a necessary step because, if a stationary equilibrium fails to exist, the number of active firms and the average productivity keep changing over time, and the integrals (2.11) and (2.12) cannot be evaluated with standard methods. To prove existence and uniqueness of a stationary equilibrium, I use the argument of Dixit and Pindyck (1994, Chapter 8), and Miao (2005), and solve for the density $g(\psi)$ of the distribution $f(\psi)$ scaled by the entry rate n, such that $f(\psi) = ng(\psi)$. I restrict my attention to equilibria in which the productivity exit threshold is lower than the initial productivity draw, that is $\underline{\psi} > \psi_e$. This condition must be verified in equilibrium. Equilibria in which firms enter and immediately exit are not considered. Formal details are relegated to Appendix 2.A.3.

A stationary equilibrium is defined by a distribution of productivity levels $f^*(\psi)$, an exit threshold R_e^*, an entry rate n^*, and an amount of liquid assets $\overline{M}^*$ such that:

1. Firms set their prices according to (2.7),
2. R_e^* is chosen to maximize the value of equity,
3. the entry condition (2.22) is satisfied,
4. the distribution $f^*(\psi)$ is an invariant measure over the interval $\psi \in [\psi_e^*, \infty)$,
5. each firm holds cash reserves equal to $M = \overline{M}^*$.

When the entry rate n^* and the distribution $f^*(\psi) = n^* g^*(\psi)$ are determined, the number of active firms and the average productivity in equilibrium can be found as $N^* = n^* \int_{\psi_e^*}^{\infty} g^*(\psi)\, d\psi$ and $\psi_A^* = \left[n^* \int_{\psi_e^*}^{\infty} \psi^{\eta-1} g^*(\psi)\, d\psi\right]^{\frac{1}{\eta-1}}$, respectively. Point 1 above says that firms set the optimal market-clearing prices, point 2 means that they choose the exit time to maximize equity-holders' value, while point 3 is the equilibrium entry condition. These conditions are standard requirements in competitive equilibrium models. In addition, point 4 is a consequence of the assumed law of large numbers while point 5 is the liquidity requirement.

The next proposition establishes existence and uniqueness of the stationary equilibrium.

Proposition 2.1 *Assume that*

$$r + \lambda - \widetilde{\mu}(\eta) > 0, \tag{2.23}$$

$$\lambda + \mu - \sigma^2 > 0, \tag{2.24}$$

$$\eta + \frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\lambda}}{\sigma^2} < 0. \tag{2.25}$$

Then, there exists a unique stationary equilibrium such that $\underline{\psi} > \psi_e^*$.

As shown in Section 2.3.2, condition (2.23) serves to keep bounded the discounted stream of expected profit and therefore the value of the active firm. Furthermore in the proof of the proposition (Appendix 2.A.2) it is shown that condition (2.24) is necessary for the existence of the stationary distribution $f^*(\psi)$ while condition (2.25) guarantees that some higher moments of the stationary distribution are finite. It is important to notice that conditions (2.23) and (2.24) impose an upper limit, call it $\overline{\eta}$, on the elasticity of substitution. Thus, a stationary equilibrium can be defined in the range $\eta \in (1, \overline{\eta}]$. Although this potentially limits the generality of the results, for example the model cannot predicts what happens in perfect competition, in Section 2.3.3 I claim the main intuition of the model has a general validity and can be easily extended for larger values of the elasticity of substitution.

2.3.2 Securities valuation

As a preliminary step, I first find the solution for the value of an unconstrained and unlevered firm, defined in (2.10).

Proposition 2.2 *The value of the unconstrained and unlevered firm is equal to*

$$\begin{aligned} V_u(R) &= (1-\tau)\left[\frac{R}{r+\lambda-\widetilde{\mu}(\eta)} - \frac{F}{r+\lambda}\right] + \\ &\quad + (1-\tau)\left[\frac{F}{r+\lambda} - \frac{R_u}{r+\lambda-\widetilde{\mu}(\eta)}\right]\left(\frac{R}{R_u}\right)^{\widetilde{\beta}(\eta)}, \quad (2.26) \end{aligned}$$

where

$$\widetilde{\beta}(\eta) = \frac{1}{2} - \frac{\widetilde{\mu}(\eta)}{\widetilde{\sigma}(\eta)^2} - \sqrt{\left[\frac{\widetilde{\mu}(\eta)}{\widetilde{\sigma}(\eta)^2} - \frac{1}{2}\right]^2 + 2\frac{r+\lambda}{\widetilde{\sigma}(\eta)^2}} < 0, \tag{2.27}$$

and

$$R_u = \frac{\widetilde{\beta}(\eta)}{\widetilde{\beta}(\eta)-1}\frac{r+\lambda-\widetilde{\mu}(\eta)}{r+\lambda}F \tag{2.28}$$

is the level of revenue that triggers exit.

Then, the proposition below defines the value of equity and debt for the levered firm.

Proposition 2.3 *The value of equity of a levered firm is equal to:*

$$\begin{aligned} E(R) &= (1-\tau)\left[\frac{R}{r+\lambda-\widetilde{\mu}(\eta)}-\frac{F+b}{r+\lambda}\right] \\ &\quad +(1-\tau)\left[\frac{F+b}{r+\lambda}-\frac{R_e^*}{r+\lambda-\widetilde{\mu}(\eta)}\right]\left(\frac{R}{R_e^*}\right)^{\widetilde{\beta}(\eta)}, \end{aligned} \tag{2.29}$$

where

$$R_e^* = \frac{\widetilde{\beta}(\eta)}{\widetilde{\beta}(\eta)-1}\frac{r+\lambda-\widetilde{\mu}(\eta)}{r+\lambda}(F+b). \tag{2.30}$$

The value of debt is equal to

$$DBT(R) = \frac{b}{r+\lambda}+\left[(1-\varepsilon)V_u-\frac{b}{r+\lambda}\right]\left(\frac{R}{R_e^*}\right)^{\widetilde{\beta}(\eta)}. \tag{2.31}$$

Firms default as soon as revenue falls below R_e^*.

2.3.3 Cash policy

Finally, here I provide the expression for the optimal amount of cash reserves. Define

$$\Omega = \max\left[0,\ 1-\frac{r+\lambda-\widetilde{\mu}(\eta)}{r+\lambda}\frac{\widetilde{\beta}(\eta)}{\widetilde{\beta}(\eta)-1}\right]. \tag{2.32}$$

The following proposition holds.

Proposition 2.4 *The optimal amount of cash is equal to:*

$$\overline{M}^* = \frac{1-\tau}{r}(F+b)\,\Omega. \tag{2.33}$$

Proposition 2.4 shows that the optimal level of cash holdings is defined by the product of two components. The first, $(1-\tau)(F+b)/r$, is a cost component and is given by the discounted stream of fixed payments. The second, Ω, is the proportion by which revenue should fall below the fixed per-period payments to induce a firm to exit the industry (this is shown in Appendix 2.A.5). This term is a proxy for firms' willingness to absorb losses in bad times and can be interpreted an indicator of how valuable is the option to remain active in the market. For this reason, Ω will be called "option component". The option component, and therefore also the optimal cash, is zero if $\widetilde{\mu}(\eta) \leq (r+\lambda)/\widetilde{\beta}(\eta) < 0$. This happens because, when the proportional growth rate $\widetilde{\mu}(\eta)$ is negative and sufficiently low, the expected fall in profitability is so rapid that that exit should occur when the firm is still earning positive profits. Such a firm never experiences losses during its (presumably short) existence and, therefore, it has no reason to hold liquid assets. Expression (2.33) has an intuitive interpretation. Since firms use liquid asset to cover losses, the optimal amount of cash depends on the expected stream of costs, the cost component, and on the willingness to cover these costs in bad times, the option component. Higher fixed costs imply larger losses when profitability is low and require a larger amount of liquidity to keep the firm alive. A larger value for the option component implies that a firm is willing to absorb greater losses before declaring default and, for this reason, it needs more cash reserves as a buffer against future cash shortfalls.

The expression for $\overline{M}^*$ is independent of the equilibrium variables N^* and ψ_A^*. This greatly simplifies the comparative statics analysis because indirect equilibrium effects do not need to be taken into account.[9] The next proposition shows the effect of competition on the optimal amount of cash.

Proposition 2.5 *If $\Omega > 0$, the optimal amount of cash $\overline{M}^*$ is increasing in η.*

Proposition (2.5) says that, if it the firm holds a strictly positive amount cash (i.e., if $\Omega > 0$), $\overline{M}^*$ increases with the intensity of competition. Since $\overline{M}^*$ is independent of the equilibrium variables and the fixed per-period payments F and b are exogenous, this effect is entirely driven by the option component Ω. Thus, the result of Proposition (2.5) is a consequence of the fact that competition makes the option to stay active in the market more valuable, reinforcing the precautionary motive for holding cash.

Equation (2.32) reveals that the elasticity of substitution η affects Ω through both the revenue volatility $\widetilde{\sigma}(\eta)$ and growth rate $\widetilde{\mu}(\eta)$. Since $\widetilde{\sigma}(\eta)$

[9] In the next section I show that, when the firm chooses its optimal capital structure, indirect equilibrium effects enter the cost component through the choice of the coupon payment b.

is increasing in η and $\partial\widetilde{\beta}(\eta)/\partial\widetilde{\sigma}(\eta) > 0$ (i.e. $\widetilde{\beta}(\eta)$ decreases in absolute value) the volatility channel has an unambiguous upward effect on Ω and, therefore, on the optimal amount of cash holdings. In contrast, the effect through the growth rate is ambiguous and positive when $\widetilde{\mu}(\eta)$ increases with the elasticity of substitution, and the other way around.[10] Both results are intuitive. A higher volatility triggers the standard real options effect. When volatility is large, business conditions can rapidly improve and this induces a firm to delay the exit decision. Similarly, a higher growth rate means that it is optimal to remain active even when current losses are sizeable. Both effects imply that firms are willing to absorb larger losses when current profitability is low. In order to implement this policy, firms need a larger amount of cash reserves.

As mentioned in Section 2.2.4, competition can both increase and decrease the revenue growth rate. The fact that the option component, and therefore the optimal amount of cash, is increasing with competition stresses the importance of the volatility channel. A higher volatility increases the amount of cash reserves even when competition has a depressing effect on the revenue growth rate. That is, although a higher degree of product substitutability may lower the growth potential, the consequent increase in volatility implies that the firm is nevertheless eager to stay longer in the market and, therefore, holds a larger amount of cash. In this sense, the increase in volatility is the key factor to explain the effect of competition on cash holdings.

As pointed at the end of Section 2.3.1, the parametric restrictions (2.23) and (2.25) imply that a stationary equilibrium does not exist for values of the product substitutability larger than $\overline{\eta}$. This confines the analysis to imperfectly competitive markets with a relatively low intensity of competition. The intuition behind Proposition 2.5, however, can be easily applied to highly competitive markets and even extended to the limit case of perfect competition. In fact, the main mechanism identified by the model is straightforward. By raising profit volatility, competition increases the value of the option to remain active in the market and strengthens the precautionary motive for holding cash. Since the effect of competition on volatility is monotonic in the range $\eta \in (1, \infty)$ (cf. the

[10] It can be checked that $\partial\Omega/\partial\widetilde{\mu}(\eta) > 0$ holds if condition (2.24) is satisfied.

definition of $\widetilde{\sigma}(\eta)$ in (2.18)), intuition suggests that the same reasoning can apply for $\eta \geq \overline{\eta}$. At the same time, however, the fact that competition increases the option value to remain active in the market can be difficult to reconcile with the idea that under perfect competition economic profits are zero. The two effects are not necessarily in contrast. If it is true that perfect competition implies zero economic profits in a static model with symmetric firms, this is not necessarily the case in a dynamic setting in which a certain degree of asymmetry is allowed. Under perfect competition, a firm that becomes more efficient than the pool of incumbents can capture the entire demand and make positive profits by setting a price just below the marginal cost of its most efficient competitor. As illustrative example, it is useful to think of a Bertrand duopoly setting. If firms are symmetric, it is well known that Bertrand interaction yields the competitive market outcome and economic profits are indeed zero. But when firms are asymmetric, that is they have different marginal costs, the most efficient one can capture the entire market and make positive profit. If we allow for idiosyncratic shocks to marginal costs, the identity of the most efficient firm can change over time and, in fact, firms may switch from having no market to capturing the entire demand. Consistent with this intuition, the model implies that under perfect competition, volatility is infinite and a firm with a productivity above the market average, $\psi > \psi_A$, enjoys infinite profits, $\lim_{\eta \to \infty} \pi = \infty$. This suggests that the argument that competition increases the option value to remain in the market does not need to be restricted in the range $\eta \in (1, \overline{\eta}]$ but can have a more general validity.

2.4 Optimal capital structure

In the previous section the coupon payment is taken as exogenous. Here, I give the full characterization of the model and let the firms to optimally choose their capital structure. The coupon payment b is chosen by the internal equity-holders at the time of entry, but before the draw of the initial productivity ψ_0 (see the timeline in Figure (2.1)). Then, firms enter the market with an amount of cash reserves sufficient to avoid inefficient liquidation and follow the optimal unconstrained policy afterwards. To obtain closed-form solutions, I further assume that firms have no fixed costs of production, i.e. $F = 0$. It follows that the optimal amount of cash is equal to

$$\overline{M} = (1-\tau)\,\frac{b}{r}\Omega, \tag{2.34}$$

(cf. equation (2.33)). Equation (2.34) implies that a firm which issues more debt must also hold more cash to meet future coupon payments. This is a "mechanical" consequence of the fact that firms hold cash to cover costs and suggests that it may exist a positive relation between leverage and liquidity. In contrast, in Section (2.4.2) I show that the endogenous determination of the capital structure implies that factors that tend to increase cash exert, in general, a downward pressure on debt, and this can generate a negative relation between cash and leverage.

Consider equation (2.20) and let all the relevant variables to be explicitly dependent on the coupon b. For convenience, the dependence on other variables is omitted. The value for the internal equity-holders is:

$$(1-\omega)\,E_{-1}(\,b) = V_{-1}(b) - L - I - \;\overline{M}\,(b)\,. \tag{2.35}$$

The optimal coupon is chosen to maximize (2.35) and satisfies the first order condition

$$V_{-1}'(b) - \;\overline{M}'\,(b) = V_{-1}'(b) - \;\frac{(1-\tau)}{r}\Omega = 0. \tag{2.36}$$

With a free access to the capital market, an unconstrained firm does not need reserves of liquidity and optimally saves on the cost of entry by raising no cash at the initial date, $M_0 = 0$. Thus, the unconstrained first-best coupon satisfies the first order condition $V_{-1}'(b) = 0$ and equalizes the marginal benefits of the tax-shield on profits with the marginal cost of an increased bankruptcy risk. In contrast, a constrained firm needs cash to survive once active in the market. But, as shown in equation (2.22), hoarding cash at the initial date involves a liquidity cost because it raises the total costs of entry. Thus, a constrained firm will also take into account that higher debt payments require a larger amount of cash (via the cost component) and increase the liquidity cost. In the first order condition (2.36), the liquidity cost is captured by the term $\overline{M}'\,(b) = (1-\tau)\,\Omega/r$, which represents the additional amount of cash that a firm should raise if debt payments increase by one unit.

2.4.1 Equilibrium

A stationary equilibrium with endogenous capital structure is defined by a productivity distribution $f^*(\psi)$, an exit threshold R_e^*, an entry rate n^*, a coupon b^* and an amount of cash $\overline{M}^*$ such that:

1. Firms set their prices according to (2.7),
2. R_e^* is chosen to maximize the value of equity,
3. the entry condition (2.22) is satisfied,
4. the distribution $f^*(\psi)$ is an invariant measure over the interval $\psi \in [\psi_e^*, \infty)$,
5. the optimal coupon b^* satisfies (2.36),
6. each firm holds cash reserves equal to $M = \overline{M}^*$.

Proposition 2.6 *If assumptions (2.23)-(2.25) hold, there exists a unique stationary equilibrium such that $\underline{\psi} > \psi_e^*$.*

2.4.2 Securities valuation and cash policy

In absence of fixed cost of production, an unlevered firm does not face any fixed payment and, therefore, it never exits. It follows that its value is simply given by the discounted stream of revenue:

$$V_u(R) = \frac{(1-\tau)}{r+\lambda-\widetilde{\mu}(\eta)} R. \tag{2.37}$$

The following proposition characterizes the equilibrium for the model with endogenous leverage.

Proposition 2.7 *The expressions for equity, debt, optimal coupon, exit threshold and cash are given by*

$$\begin{aligned} E(R) &= (1-\tau)\left[\frac{R}{r+\lambda-\widetilde{\mu}(\eta)} - \frac{b^*}{r+\lambda}\right] \\ &\quad + (1-\tau)\left[\frac{b^*}{r+\lambda} - \frac{R_e^*}{r+\lambda-\widetilde{\mu}(\eta)}\right]\left(\frac{R}{R_e^*}\right)^{\widetilde{\beta}(\eta)}, \end{aligned} \tag{2.38}$$

$$DBT\left(R\right)=\frac{b^{*}}{r+\lambda}+\left[\left(1-\varepsilon\right)V_{u}-\frac{b^{*}}{r+\lambda}\right]\left(\frac{R}{R_{e}^{*}}\right)^{\widetilde{\beta}(\eta)}, \tag{2.39}$$

$$b^{*}=\max\left[0,\ b_{u}^{*}\left(1-\frac{1-\tau}{\tau}\Omega\right)^{-\frac{1}{\widetilde{\beta}(\eta)}}\right] \tag{2.40}$$

$$R_{e}^{*}=\frac{\widetilde{\beta}\left(\eta\right)}{\widetilde{\beta}\left(\eta\right)-1}\frac{r+\lambda-\widetilde{\mu}\left(\eta\right)}{r+\lambda}b^{*}, \tag{2.41}$$

$$\overline{M}^{*}=\left(1-\tau\right)\frac{b^{*}}{r}\Omega, \tag{2.42}$$

where

$$b_{u}^{*}=\frac{r+\lambda}{r+\lambda-\widetilde{\mu}\left(\eta\right)}\frac{\widetilde{\beta}\left(\eta\right)\ -1}{\widetilde{\beta}\left(\eta\right)}\Gamma\left[1-\widetilde{\beta}\left(\eta\right)-\varepsilon\widetilde{\beta}\left(\eta\right)\frac{\left(1-\tau\right)}{\tau}\right]^{\frac{1}{\widetilde{\beta}(\eta)}} \tag{2.43}$$

is the optimal coupon for an unconstrained firm, and

$$\Gamma=\frac{\left(\ \psi_{A}^{*}\right)^{1-\eta}}{\eta N^{*}}\left\{\frac{\overline{\psi}^{(\eta-1)\widetilde{\beta}(\eta)+1}-\underline{\psi}^{(\eta-1)\widetilde{\beta}(\eta)+1}}{\left[\left(\eta-1\right)\widetilde{\beta}\left(\eta\right)+1\right]\left(\overline{\psi}-\underline{\psi}\right)}\right\}^{\frac{1}{\widetilde{\beta}(\eta)}}. \tag{2.44}$$

The total value of the firm is given the sum of equity and debt and can be written as the value of the unlevered firm, plus the tax benefit of debt, plus the risk-adjusted bankruptcy cost, as determined by the fraction of the unlevered firm's value lost at default:

$$V(R)=\frac{\left(1-\tau\right)}{r+\lambda-\widetilde{\mu}\left(\eta\right)}R+\frac{\tau b^{*}}{r+\lambda}\left[1-\left(\frac{R}{R_{e}^{*}}\right)^{\widetilde{\beta}(\eta)}\right]+\varepsilon V_{u}\left(R_{e}^{*}\right)\left(\frac{R}{R_{e}^{*}}\right)^{\widetilde{\beta}(\eta)}. \tag{2.45}$$

Equation (2.40) shows that the optimal coupon payment for a constrained firm is lower than or equal to the one of an unconstrained and cashless firm. This happens because, as explained in the discussion following equation (2.36), debt imposes an additional liquidity cost to the constrained firm.[11] Furthermore, the wedge between b_u^* and b^* increases with the option component Ω. The larger is the option value to remain in the market,

[11]Equation (2.40) also shows that, if $\tau \leq \Omega/(1+\Omega)$, a constrained firm issues no debt and would rather prefer to become a net lender (i.e. $b^* < 0$, a possibility ruled out by assumption). This contrasts with the case of an unconstrained firm for which it is always optimal to issue debt in the presence of a positive tax-rate (indeed, it holds that $\lim_{\tau\to 0} b_u^* = 0$).

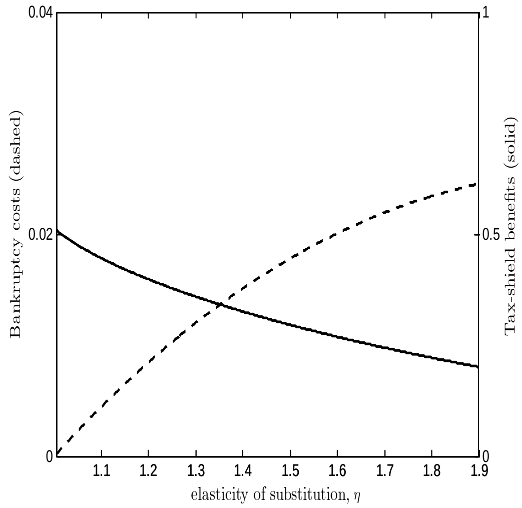

Figure 2.2: The effect of the elasticity of substitution on the bankruptcy costs (dashed line) and tax benefits of debt (solid line). Parameter values: $r = 0.04$, $\mu = 0$, $\sigma = 0.1$, $\lambda = 0.05$, $\overline{\psi} = 5$, $\underline{\psi} = 1$, $\varepsilon = 0.8$, $I = 1$. $L = 0$. All graphs are plotted for a range of values of η such that conditions (2.23) and (2.25) are satisfied.

the more compelling is the need to avoid default, and the stronger is the incentive to reduce debt payments compared to the first best. Interestingly, this mechanism may also give rise a to a negative relation between cash and leverage. When the value to remain active in the market is large, firms are willing to increase their cash balance to be able to withstand negative shocks. But at the same time, they have also an incentive to reduce the risk of liquidation by lowering fixed interest payments on debt. Therefore, factors that increase cash reserves tend to have the opposite effect on leverage.

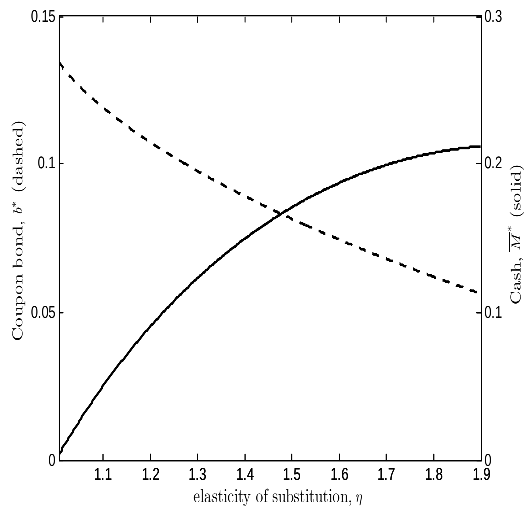

Figure 2.3: The effect of competition on the coupon bond and optimal amount of cash. Parameter values: $r = 0.05$, $\mu = 0$, $\sigma = 0.1$, $\lambda = 0.05$, $\overline{\psi} = 5$, $\underline{\psi} = 1$, $\varepsilon = 0.8$, $I = 1$. $L = 0$.

2.4.3 Model analysis

To illustrate the predictions of the model, I set the parameter values as follows.[12] The risk-free interest rate is equal to $r = 0.05$ and the productivity growth rate is $\mu = 0$. Miao (2005) sets the exogenous Poisson death rate equal to 0.04 based on the consideration that the turnover rate is approximately 7% (Dunne et al. (1988)) and that the default rate is around 3% (Brady and Bos (2002)). Consistent with this evidence, I set $\lambda = 0.04$. The entry and issuance costs are normalized to one and zero, respectively, $I = 1$ and $L = 0$. The corporate tax rate is set equal to $\tau = 0.34$ and the recovery rate equal to $(1 - \varepsilon) = 0.2$, as in Miao (2005). Finally, a value must be chosen for the upper and lower bounds of the initial productivity level, $\overline{\psi}$ and $\underline{\psi}$. Since possible reference values to match (as, for example, Tobin's q for the average firm or the turnover rate) are insensitive to the parameterization of $\overline{\psi}$ and $\underline{\psi}$, the choice is arbitrary. I set $\overline{\psi} = 5$ and $\underline{\psi} = 1$. Finally, the range of values for η in the comparative statics analysis must satisfy assumptions (2.23) and (2.25).

The analysis of Section 2.3 revealed that a higher elasticity of substitution makes the option to stay active in the market more valuable and increases the optimal amount of cash reserves. Now, the overall effect also depends on how competition affects the choice of the coupon payment b^*. Let me consider, first, the effects of competition on costs and benefits of issuing debt. As implied by equation (2.36), the optimal capital structure decision is the result of a trade-off between the tax benefits and the total costs of leverage, given by the increased risk-adjusted bankruptcy cost and the higher liquidity cost. Figure 2.2 shows that competition reduces the tax-shield benefits of debt and increases the risk-adjusted costs of bankruptcy. Although the driving forces behind this result are difficult to pin down analytically, the explanation is nevertheless intuitive. Competition decreases the expected profits for the average firm and, at the same time, increases volatility. Lower profits and higher volatility together raise the risk of default, lower the benefits of the tax-shield on profits, and increase the risk-adjusted bankruptcy costs. Furthermore, since a higher elasticity of substitution has an positive effect on the option component Ω, debt

[12] For different parameterizations, the qualitative results of the model do not change.

η	**Cash ratio**	**Leverage ratio**
1	0	0.464
1.1	0.016	0.425
1.2	0.029	0.393
1.3	0.041	0.364
1.4	0.051	0.339
1.5	0.059	0.313
1.6	0.065	0.289
1.7	0.070	0.267
1.8	0.074	0.244
1.9	0.075	0.222

Table 2.1: The effect of the elasticity of substitution on the cash and leverage ratios for the average firm. Parameter values: $r = 0.05, \mu = 0, \sigma = 0.1, r = 0.05, \psi = \psi_A, \overline{\psi} = 5, \underline{\psi} = 1, \varepsilon = 0.8, I = 1, L = 0$. For $\eta = 1$, the value of the firm is infinite and the equilibrium problem has no solution. The reported values for the cash and leverage ratios are to be intended for η approaching one from above.

payments must be backed by a larger amount of cash, so that the liquidity cost of debt also increases.[13]

The above discussion implies that the net benefits of leverage are unequivocally lower when competition is more intense. For this reason, the optimal coupon b^* is expected to decrease with the elasticity of substitution. Figure 2.3 (dashed line) confirms this intuition and shows that b^* (and thus the cost component) is indeed a monotonically decreasing function of η. Reminding the result of Section 2.3, this implies that two contrasting forces determine the effect of competition on the optimal amount of cash reserves. On the one hand, competition makes firms' profits more volatile increasing the option value to remain active in the market. This force has a positive effect on cash. But on the other hand, competition increases the risk of default and induces the firms to adjust their cost structure by reducing debt payments. Lower debt payments decrease fixed costs and tend to reduce the optimal amount of cash re-

[13] Indeed, from equation (2.36), note that $\overline{M}'(b) = (1 - \tau)\,\Omega/r \geq 0$.

serves. The interaction of these two forces suggests that, potentially, cash can both increase or decrease with competition depending on whether the effect on the option component or the one on the cost component dominates. Figure 2.3 (solid line) shows that, in the numerical example, $\overline{M}^*$ increases monotonically with η. The existence of a region in which the effect on the cost component is dominant, and competition has a negative effect on cash, cannot be ruled out with certainty. However, I was unable to find parameter values such that $\overline{M}^*$ is decreasing in η and, at the same time, conditions (2.23)-(2.25) and the requirement $\underline{\psi} > \psi_e^*$ are satisfied.

Beside the simulation results, other motivations support the idea that the downward effect via the cost component is likely to be of second order. To begin with, cash holdings surely increase with competition for low levels of product substitutability. To see this, consider the limit case in which there is no substitutability between products. Recalling the definition of $\widetilde{\mu}(\eta)$ and $\widetilde{\beta}(\eta)$, it can be shown that $\lim_{\eta \to 1} \overline{M}^* = 0$. The explanation is that, with no product substitutability, firms are insulated against stochastic fluctuations and have no reason to hold cash.[14] An increase in the degree of product substitutability has the effect of introducing uncertainty and generates the need for liquidity. Therefore, cash reserves are increasing when the intensity of competition is initially low. Furthermore, to find closed-form solutions, I solved the capital structure model by setting the fixed production costs equal to zero, $F = 0$. However, the latter assumption is hardly realistic. With positive fixed cost of production, interest payments on debt represent a smaller fraction of the total fixed payments. Then, the reduction in leverage induced by competition has a relatively lower impact on the total size of the cost component and the upward effect on the option value is more likely to prevail. Finally, I assume that firms freely choose their capital structure before the entry date and, thus, have a high degree of flexibility in adjusting debt payments to the market structure. However, when capital markets are imperfect, it

[14] Indeed, from (2.15) and (2.9) it follows that $\lim_{\eta \to 1} \pi(R) = (1-\tau)(1/\eta N - F - b)$. The intuition is that, when the elasticity of substitution is equal to one, an increase (decrease) in price is perfectly offset by a corresponding decrease (increase) in demand, so that profits remain unaffected. The cost of productivity fluctuations is entirely borne by consumers.

can be difficult for the firms to substantially reduce their debt payments. This may be because access to equity financing can be even more costly than debt or because the original level of leverage is already low (e.g., financially constrained firms are often zero-debt firms). Lack of financial flexibility impairs the functioning of the cost component channel and makes the upward effect on cash due to the option component more likely to dominate.

To establish a closer link with the empirical literature, I present the prediction of the model for two widely used measures of cash and leverage. Table 2.1 shows the effects of the elasticity of substitution on the leverage ratio (defined as $DBT(R)/(V(R)+\overline{M}^*))$ and cash ratio (defined as and $\overline{M}^*/(V(R)+\overline{M}^*))$ for the average firm in the industry, that is a firm with productivity equal to $\psi = \psi_A$. The table reveals a pattern analogous to Figure 2.3 and shows that the leverage ratio decreases with the intensity of competition while the cash ratio increases. As already anticipated, the key mechanism identified by the model is as follows. Competition, mainly due the effect on profit volatility, increases the option value to remain active in the market and makes firms more willing to hold cash to withstand negative shocks. At the same time, a larger option value is an incentive to lower the risk of liquidation, and firms do so by reducing leverage and interest payments on debt. This prediction is the main message of the model and finds support in the preliminary evidence, reported in Frésard and Valta (2011), that firms increase cash reserves and substitute debt for equity in response to increased competitive pressure.

From a quantitative point of view, Bates et al. (2009) show that in recent years the cash ratio reached values up to 0.15 for the median firm and well above 0.2 for the average firm. Despite the fact that in the model firms hold an amount of cash large enough to avoid inefficient liquidation, the predicted cash ratio for the average firm is below the values observed in the data (see Table 2.1). This result is easily explained by the assumption of zero fixed production costs. Positive fixed costs of production would increase the cost component and bring the cash ratio close to the empirically observed values. Importantly, the model predicts a leverage ratio close to the historical average of 25% reported in Barclay et al. (2006). A known concern with the standard contingent claim models of optimal capital structure is that they usually predict unreasonably high values for the leverage ratio. However, as pointed out by Miao (2005), in a stationary equilibrium model with heterogeneous firms, there are not many incumbents with a low value of equity because the less efficient competitors are forced out of the market. Self-selection implies that the average firm will result relatively less leveraged. Beside such equilibrium mechanism, in the model an additional effect is at work. Due to the need of raising cash, constrained firms face a higher cost of leverage and choose a more debt-conservative capital structure (see equation (2.40) and its interpretation). This further decreases the leverage ratio of the typical firm.

2.5 Conclusion

This work studies the effects of competition on firms' cash holdings. I develop a model of monopolistic competition with imperfect capital markets in which firms make entry, exit and pricing decisions, choose their capital structure, and hoard cash to avoid inefficient liquidation. Product market competition affects firms' cash policy in two different and contrasting ways. On the one hand, by increasing profit volatility and the option value to remain active in the market, it reinforces the precautionary motive for holding cash. But on the other hand, by increasing the risk of bankruptcy, it induces the firms to reduce leverage. With a more debt-conservative capital structure firms face lower interest payments on debt and need less cash to avoid inefficient liquidation. Although the overall effect on cash is potentially ambiguous, the upward effect due to a larger option value dominates.

The model generates the following predictions. First, cash holdings increase with competition. Second, there is a negative relation between cash holdings and leverage induced by the option value to remain active in the market. When this option value is larger, firms are willing to hold more cash to withstand negative shocks. At the same time, firms face a more compelling need to limit losses in bad times and they do so by reducing leverage and interest payments on debt. Finally, the model also predicts that, by increasing the option value, competition exacerbates the negative relation between cash and debt.

2.A Appendix

2.A.1 Valuation problem in terms of ψ

For future reference, it is useful to express the value of equity and the exit threshold in terms of ψ. The value of equity $E(\psi)$ satisfies

$$\frac{1}{2}\sigma^2\psi^2 E''(\psi) + \mu\psi E'(\psi) - (r+\lambda)E(\psi) + \frac{1}{\eta N}\left(\frac{\psi}{\psi_A}\right)^{\eta-1} - F - b = 0, \tag{2.46}$$

subject to

$$\lim_{\psi\to\infty} E(\psi) = \frac{1}{r+\lambda-\widetilde{\mu}(\eta)}\frac{1}{\eta N}\left(\frac{\psi}{\psi_A}\right)^{\eta-1} - \frac{F+b}{r+\lambda}, \tag{2.47}$$

$$E(\psi_e) = 0, \tag{2.48}$$

and

$$E'(\psi_e) = 0. \tag{2.49}$$

Equation (2.47) means that, when ψ grows larger, the probability of exit becomes negligible and the firm's value is simply given by the discounted stream of profits. Equation (2.48) implies that at exit time the value of the firm is zero, while equation (2.49) means that the exit threshold is chosen to maximize equity-holders' value. Solving (2.46) under (2.47)-(2.49) yields the expressions for the exit threshold ψ_e and equity $E(\psi)$:

$$\psi_e = \psi_A\left[\frac{\beta}{\beta-(\eta-1)}\frac{r+\lambda-\widetilde{\mu}(\eta)}{r+\lambda}\eta N(F+b)\right]^{\frac{1}{\eta-1}}, \tag{2.50}$$

$$\begin{aligned} E(\psi) &= \left[\frac{1}{r+\lambda-\widetilde{\mu}(\eta)}\frac{1}{\eta N}\left(\frac{\psi}{\psi_A}\right)^{\eta-1} - \frac{F+b}{r+\lambda}\right] + \\ &+ \left[\frac{F+b}{r+\lambda} - \frac{1}{r+\lambda-\widetilde{\mu}(\eta)}\frac{1}{\eta N}\left(\frac{\psi_e}{\psi_A}\right)^{\eta-1}\right]\left(\frac{\psi}{\psi_e}\right)^{\beta} \end{aligned} \tag{2.51}$$

where $\beta = 1/2 - \alpha/\sigma^2 - \sqrt{[\alpha/\sigma^2 - 1/2]^2 + 2r/\varsigma^2} < 0$. Notice that condition (2.23) guarantees that the value of equity is bounded.

2.A.2 Proof of Proposition 2.1

Here I prove existence and uniqueness of the stationary equilibrium with exogenous leverage. To find the productivity distribution of incumbent firms, I solve for the density $g^*(\psi)$ of the distribution $f^*(\psi)$ scaled by the entry rate n, such that $f(\psi) = ng(\psi)$. Overall, I have four unknowns $g^*(\psi)$, ψ_e^*, n^*, M^*.

I proceed according to the following steps. First, I find the value for $W = N(\psi_A)^{\eta-1}$ which satisfies the entry condition. Using equation (2.50), this allows determining the unique equilibrium exit threshold ψ_e^*. Then, I find the scaled density $g^*(\psi)$ and, using equation (2.22), the equilibrium entry rate n^*. Finally, the number of active firms is determined as $N^* = n^* \int_{\psi_e^*}^{\infty} g^*(\psi)\, d\psi$.

Let me find, first, the exit threshold ψ_e^*. Consider the expected value of the firm before entry:

$$V_{-1} = \mathbb{E}_{-1}\left\{ \frac{R}{r+\lambda-\widetilde{\mu}(\eta)} - \frac{F}{r+\lambda} + \left[\frac{F}{r+\lambda} - \frac{R_e}{r+\lambda-\widetilde{\mu}(\eta)}\right]\left(\frac{R}{R_e}\right)^{\widetilde{\beta}(\eta)} \right\}. \tag{2.52}$$

Using the definition of R and computing the expected values, the expression in (2.52) can be rewritten as

$$\begin{aligned} V_{-1} \quad = \quad & \left[\frac{1}{r+\lambda-\widetilde{\mu}(\eta)}\frac{\Psi_1}{\eta W} - \frac{F}{r+\lambda}\right] + \\ & + \left[\frac{F}{r+\lambda} - \frac{R_e}{r+\lambda-\widetilde{\mu}(\eta)}\right]\left(\frac{1}{R_e \eta W}\frac{1}{r+\lambda-\widetilde{\mu}(\eta)}\right)^{\widetilde{\beta}(\eta)} \Psi_2, \end{aligned} \tag{2.53}$$

where $W = N(\psi_A)^{\eta-1}$, while

$$\Psi_1 = \frac{\overline{\psi}^{\eta} - \underline{\psi}^{\eta}}{\eta\left(\overline{\psi} - \underline{\psi}\right)} \tag{2.54}$$

and

$$\Psi_2 = \frac{\overline{\psi}^{(\eta-1)\widetilde{\beta}(\eta)+1} - \underline{\psi}^{(\eta-1)\widetilde{\beta}(\eta)+1}}{\left[(\eta-1)\widetilde{\beta}(\eta)+1\right]\left(\overline{\psi} - \underline{\psi}\right)} \tag{2.55}$$

are the $(\eta-1)\,th$ and $(\eta-1)\,\widetilde{\beta}(\eta)\,th$ moments of the initial productivity draw. V_{-1} decreases monotonically in W and, therefore, there is a unique combination $W^* = N^* (\psi_A^*)^{\eta-1}$ which satisfies (2.22). Once W^* is determined, using equation (2.50), the solution for the unique equilibrium exit threshold can be found:

$$\psi_e^* = \left[\frac{\beta}{\beta-(\eta-1)}\frac{r+\lambda-\widetilde{\alpha}(\eta)}{r+\lambda}\eta W^* F\right]^{\frac{1}{\eta-1}}. \tag{2.56}$$

The equilibrium exit threshold (2.56) uniquely defines the support $\psi \in [\psi_e^*, \infty)$ for the productivity distribution of the active firms. I now solve for the density $g(\psi)$ of the distribution $f(\psi)$ scaled by the entry rate n. This step of the proof follows Dixit and Pindyck (1994, Chapter 8) and Miao (2005).

I start using the transformation $z = \log(\psi)$. The new variable z follows a geometric Brownian motion:

$$\frac{dz_t}{z_t} = \alpha dt + \varsigma dW_t, \tag{2.57}$$

where $\alpha = \mu - 1/2\sigma^2$ and $\varsigma = \sigma$. The initial productivity draw in terms of z has an exponential distribution over $[\underline{z}, \overline{z}]$, where $\underline{z} = \log \underline{\psi}$ and $\overline{z} = \log \overline{\psi}$. This distribution has density function

$$m(z) = e^{(z-\widetilde{z})}, \tag{2.58}$$

where $\widetilde{z} = \log\left(\overline{\psi} - \underline{\psi}\right)$.

The equilibrium density is defined over the interval $z \in [z_e^*, \infty)$, where $z_e^* = \log(\psi_e^*)$. I derive it by using a binomial approximation for the Brownian motion. Time is divided in intervals of length dt and the z-space in small segments of length $dh = \varsigma\sqrt{dt}$. In each time interval, all firms located in a segment will move away. A fraction $(1-\lambda)\,dt$ will move up or down in proportions q_u and q_d, where

$$q_u = \frac{1}{2}\left[1 + \frac{\alpha}{\varsigma}\sqrt{dt}\right] \text{ and } q_d = \frac{1}{2}\left[1 - \frac{\alpha}{\varsigma}\sqrt{dt}\right],$$

while a fraction λdt will die because of the Poisson shock. Define $n\phi(z)$ the density of active firms in a segment centered at z, where the entry rate n is a scale factor. Such a density is time invariant if, in a given z interval, the outflow of firms due to the Brownian shocks or the Poisson death is offset by the inflow of incumbents with higher or lower productivity and new entrants. Consider, first, the case where $z \in [\underline{z}, \overline{z}]$. In this segment the old firms will be replaced by new entrants and firms coming from above and below. The density in the segment remains constant over time if the rate at which firms leave equal the rate arrival rate:

$$n\phi(z)\,dh = ndtm(z)\,dh + q_d(1-\lambda dt)\,n\phi(z)\,dh + q_u(1-\lambda dt)\,n\phi(z)\,dh. \tag{2.59}$$

Applying the Taylor's expansion to the above equation and simplifying yields

$$\frac{1}{2}\varsigma^2\phi''(z) - \alpha\phi'(z) - \lambda\phi(z) + m(z) = 0. \tag{2.60}$$

Consider, now, segments centered in $z \in [\overline{z}, \infty)$ and $z \in [z_e^*, \underline{z}]$, where $z_e^* = \log\psi_e^*$. In these cases the segment is outside the range for the initial productivity draw and, therefore, there is no flow of new entrants, but only incumbent firms coming from above and below. Steps analogous as before yield the ordinary differential equation

$$\frac{1}{2}\varsigma^2\phi''(z) - \alpha\phi'(z) - \lambda\phi(z) = 0. \tag{2.61}$$

Solution of (2.60) and (2.61) implies that

$$\phi(z) = A_1e^{\gamma_1 z} + A_2e^{\gamma_2 z} \text{ in the region } z \in (\overline{z}, \infty) \tag{2.62}$$

$$\phi(z) = B_1e^{\gamma_1 z} + B_2e^{\gamma_2 z} + \frac{e^{(z-\tilde{z})}}{\lambda + \alpha - \frac{1}{2}\varsigma^2} \text{ in the region } z \in [\underline{z}, \overline{z}] \tag{2.63}$$

$$\phi(z) = C_1e^{\gamma_1 z} + C_2e^{\gamma_2 z} \text{ in the region } z \in (z_e^*, \underline{z}) \tag{2.64}$$

where

$$\gamma_1 = \frac{\alpha - \sqrt{\alpha^2 + 2\varsigma^2\lambda}}{\varsigma^2} < 0 \tag{2.65}$$

and

$$\gamma_1 = \frac{\alpha + \sqrt{\alpha^2 + 2\varsigma^2\lambda}}{\varsigma^2} > 0. \tag{2.66}$$

Constants A_1, A_2, B_1, B_2, C_1 and C_2 are determined by the boundary conditions

$$\int_{\overline{z}}^{\infty} \phi(z)\,dz < \infty \tag{2.67}$$

$$\phi(z_e^*) = 0 \tag{2.68}$$

$$\lim_{z\uparrow\overline{z}}\phi(z) = \lim_{z\downarrow\overline{z}}\phi(z) \tag{2.69}$$

$$\lim_{z\uparrow\overline{z}}\phi'(z)=\lim_{z\downarrow\overline{z}}\phi'(z) \tag{2.70}$$

$$\lim_{z\uparrow\underline{z}}\phi(z)=\lim_{z\downarrow\underline{z}}\phi(z) \tag{2.71}$$

$$\lim_{z\uparrow\underline{z}}\phi'(z)=\lim_{z\downarrow\underline{z}}\phi'(z) \tag{2.72}$$

Condition (2.67) says that the number of active firms is finite and implies that $A_2 = 0$. Condition (2.68) derives from the fact that at z_e^* firms exit. Conditions (2.69)-(2.72) ensure smoothness of the density function $\phi(z)$. Solving the system of linear equations (2.68)-(2.72), and writing the solution in in terms of ψ, yields the following expression for the coefficients A_1, B_1, B_2, C_1, C_2:

$$A_1=\frac{(1-\gamma_1)(\psi_e^*)^{\gamma_2-\gamma_1}\left(\overline{\psi}^{1-\gamma_2}-\underline{\psi}^{1-\gamma_2}\right)-(1-\gamma_2)\left(\overline{\psi}^{1-\gamma_1}-\underline{\psi}^{1-\gamma_1}\right)}{(r+\lambda-\mu)\left(\overline{\psi}-\underline{\psi}\right)(\gamma_2-\gamma_1)}, \tag{2.73}$$

$$B_1=\frac{(1-\gamma_1)(\psi_e^*)^{\gamma_2-\gamma_1}\left(\overline{\psi}^{1-\gamma_2}-\underline{\psi}^{1-\gamma_2}\right)+(1-\gamma_2)\underline{\psi}^{1-\gamma_1}}{(r+\lambda-\mu)\left(\overline{\psi}-\underline{\psi}\right)(\gamma_2-\gamma_1)} \tag{2.74}$$

$$B_2=-\frac{(1-\gamma_1)\overline{\psi}^{1-\gamma_2}}{(r+\lambda-\mu)\left(\overline{\psi}-\underline{\psi}\right)(\gamma_2-\gamma_1)} \tag{2.75}$$

$$C_1=\frac{(1-\gamma_1)(\psi_e^*)^{\gamma_2-\gamma_1}\left(\overline{\psi}^{1-\gamma_2}-\underline{\psi}^{1-\gamma_2}\right)}{(r+\lambda-\mu)\left(\overline{\psi}-\underline{\psi}\right)(\gamma_2-\gamma_1)} \tag{2.76}$$

$$C_1=\frac{(1-\gamma_1)\left(\underline{\psi}^{1-\gamma_2}-\overline{\psi}^{1-\gamma_2}\right)}{(r+\lambda-\mu)\left(\overline{\psi}-\underline{\psi}\right)(\gamma_2-\gamma_1)} \tag{2.77}$$

The solution for the scaled equilibrium density is

$$g^*(\psi)=\begin{cases} A_1\psi^{\gamma_1} \text{ if } \psi\in\left(\overline{\psi},\infty\right), \\ B_1\psi^{\gamma_1}+B_2\psi^{\gamma_2}+\frac{\psi}{\left(\overline{\psi}-\underline{\psi}\right)(\lambda+\mu-\sigma^2)} \text{ if } \psi\in\left[\underline{\psi},\overline{\psi}\right], \\ C_1\psi^{\gamma_1}+C_2\psi^{\gamma_2} \text{ if } \psi\in\left(\psi_e^*,\underline{\psi}\right). \end{cases} \tag{2.78}$$

Assumption (2.24) guarantees that in the region $\psi\in\left[\underline{\psi},\overline{\psi}\right]$ the equilibrium density is finite. For also the average productivity ψ_A^* to be finite it must be that $\int_{\psi_e^*}^{\infty}\psi^{\eta-1}g^*(\psi)\,d\psi$ is finite. It is sufficient to show that

$\int_{\underline{\psi}}^{\infty} \psi^{\eta-1} g^* (\psi)\, d\psi$ is finite. The integral is bounded if $\eta + \gamma_1 < 0$. This is true when assumption (2.25) holds.

The next step is to derive the equilibrium entry rate n^*. Having the expression for $g^*(\psi)$ the equilibrium entry rate n^* must satisfy the entry condition (2.22). The expected value of the firm before the initial productivity draw $V_{-1}(\psi)$ monotonically declines in n and, therefore, there is a unique n^* that satisfies (2.22). Then, the average productivity, the stationary distribution and the number of firms are determined by

$$\psi_A^* = \left[n^* \int_{\psi_e^*}^{\infty} \psi^{\eta-1} g^* (\psi)\, d\psi \right]^{\frac{1}{\eta-1}}, \tag{2.79}$$

$$f^* (\psi) = n^* g^* (\psi), \tag{2.80}$$

$$N^* = n^* \int_{\psi_e^*}^{\infty} g^* (\psi)\, d\psi, \tag{2.81}$$

and this completes the definition of the stationary equilibrium. ■

2.A.3 Proof of Proposition 2.2

The solution of equation (2.10) satisfies

$$\frac{1}{2} \widetilde{\sigma} (\eta)^2 R^2 V_u''(R) + \widetilde{\mu} (\eta) R V_u'(R) - (r + \lambda) V_u(R) + R - F = 0, \tag{2.82}$$

subject to

$$\lim_{R \to \infty} V(R) = \frac{R}{r + \lambda - \widetilde{\mu} (\eta)} - \frac{F}{r + \lambda}, \tag{2.83}$$

$$V_u(R_u) = 0, \tag{2.84}$$

and

$$V_u'(R_u) = 0, \tag{2.85}$$

where R_u is the level of revenue that triggers exit. Conditions (2.83)-(2.85) have the same interpretation as (2.48)-(2.49). Solving (2.82) under (2.83)-(2.85) yields (2.26) and (2.28).

2.A.4 Proof of Proposition 2.3

The value of equity can be found in a way analogous to the value of the unconstrained and unlevered firm (see the Appendix 2.A.3). The value of debt satisfies

$$\frac{1}{2}\widetilde{\sigma}(\eta)^2 R^2 DBT''(R)+\widetilde{\mu}(\eta) RDBT'(R)-(r+\lambda) DBT(R)+b=0, \quad (2.86)$$

subject to

$$\lim_{R\to\infty} DBT(R)=\frac{b}{r+\lambda}, \quad (2.87)$$

and

$$DBT(R_e)=(1-\varepsilon)\ V_u. \quad (2.88)$$

Condition (2.87) means that, when the revenue grows larger, default becomes a remote possibility and the value of debt equals its face value. Condition (2.88) means that, at default, debt-holders recoup the liquidation value. Solving (2.86) under (2.87)-(2.88) yields (2.31).

2.A.5 Proof of Proposition 2.4

Assume that the firm makes losses at the time of exit, i.e. $R_e^* - b - F < 0$. To surely avoid inefficient liquidation, the firm must be always able to meet its payments. Without access to external funds, this implies that its cash reserves must provide a stream of interest income sufficient to cover the worst-case losses under the optimal unconstrained policy. Indeed, revenue could remain arbitrarily close to the exit boundary R_e^* for an infinite time without falling below it.

Hence, the optimal amount of cash $\overline{M}^*$ satisfies

$$\int_t^\infty e^{-(\rho+\lambda)(s-t)} r\overline{M}^* ds = -(1-\tau)\int_t^\infty e^{-(\rho+\lambda)(s-t)}\left(R_e^*-b-F\right) ds. \quad (2.89)$$

If the firm makes non-negative profits at the time of exit, $R_e^* - b - F > 0$, then it will hold no cash. It follows that the expression for the optimal amount of cash is:

$$\overline{M}^* = \max\left[0,(1-\tau)\frac{b+F-R_e^*}{r}\right]. \quad (2.90)$$

Substituting equation (2.30) in (2.90) yields:

$$\overline{M}^{*} = \frac{1-\tau}{r}\left(F+b\right)\Omega, \tag{2.91}$$

where the expression for Ω is given in (2.32). Combining (2.90) and (2.91), it is immediate to see that Ω represents the proportion by which revenue must drop below the fixed payment $F+b$ to trigger default. ■

2.A.6 Proof of Proposition 2.5

To prove Proposition 2.5, I find it convenient to rewrite the optimal amount of cash using the solution procedure outlined in Appendix 2.A.1. Substituting equation (2.50) in $R_e^{*} = 1/\eta N \left(\psi_e^{*}/\psi_A\right)^{\eta-1}$, and using (2.90), the optimal amount of liquid assets can be found as

$$\overline{M}^{*} = \frac{F+b}{r}\,\widetilde{\Omega}, \tag{2.92}$$

where

$$\widetilde{\Omega} = \max\left[0,\ 1 - \frac{r+\lambda-\widetilde{\mu}\left(\eta\right)}{r+\lambda}\frac{\beta}{\beta-\left(\eta-1\right)}\right]. \tag{2.93}$$

Equations (2.32) and (2.93) allow to establish the equality:

$$\frac{\widetilde{\beta}\left(\eta\right)}{\widetilde{\beta}\left(\eta\right)-1} = \frac{\beta}{\beta-\left(\eta-1\right)}. \tag{2.94}$$

The option component $\widetilde{\Omega}$, and therefore the optimal amount of cash $\overline{M}^{*}$, strictly positive if

$$\mu > \frac{r+\lambda}{\beta} - \frac{1}{2}\left(\eta-2\right)\sigma^{2}. \tag{2.95}$$

Differentiating (2.93) with respect to η yields

$$\frac{d\widetilde{\Omega}}{d\eta} = \frac{\beta}{2\left(r+\lambda\right)r}\frac{2\left(\mu\beta-r-\lambda\right)-\left(3-2\eta\right)\beta\sigma^{2}-\left(\eta-1\right)^{2}\sigma^{2}}{\left[\beta-\left(\eta-1\right)\right]^{2}}. \tag{2.96}$$

From (2.96) I obtain that $d\widetilde{\Omega}/d\eta > 0$ and $d\overline{M}^{*}/d\eta > 0$ if $\mu > \left(r+\lambda\right)/\beta - 1/2\left(2\eta-3\right)\sigma^{2} + \left(\eta-1\right)^{2}\sigma^{2}/2\beta$, which is always true if (2.95) is satisfied. ■

2.A.7 Proof of Proposition 2.6

Substituting (2.43) in (2.50), and using (2.40) and (2.44), it can be noticed that the expression for ψ_e^* is independent of the endogenous variables ψ_A^* and N^* and, therefore, it uniquely identifies the equilibrium exit threshold. Then, existence and uniqueness of the equilibrium can be proved as in Appendix 2.A.2. ■

2.A.8 Proof of Proposition 2.7

The expressions for equity, debt and cash can be found following the same steps of the proofs of Propositions 2.2, 2.3, and 2.4. The optimal coupon b^* maximizies the expected value of the firm at the entry date V_{-1}. The expression for V_{-1} is given by

$$\begin{aligned}
V_{-1} &= \frac{(1-\tau)}{r+\lambda-\widetilde{\mu}(\eta)}\mathbb{E}[R] + \frac{\tau b^*}{r+\lambda}\left[1-\mathbb{E}\left(\frac{R}{R_e^*}\right)^{\widetilde{\beta}(\eta)}\right] + \varepsilon V_u(R_e^*)\,\mathbb{E}\left(\frac{R}{R_e^*}\right)^{\widetilde{\beta}(\eta)} \\
&= \frac{(1-\tau)}{r+\lambda-\widetilde{\mu}(\eta)}\Gamma_0 + \frac{\tau b^*}{r+\lambda}\left[1-\frac{\Gamma}{(R_e^*)^{\widetilde{\beta}(\eta)}}\right] + \varepsilon V_u(R_e^*)\frac{\Gamma}{(R_e^*)^{\widetilde{\beta}(\eta)}},
\end{aligned}$$

where $\Gamma_0 = (\psi_A^*)^{1-\eta}/\eta N^* \Psi_1^{\frac{1}{\widetilde{\beta}(\eta)}}$ and $\Gamma = (\psi_A^*)^{1-\eta}/\eta N^* \Psi_2^{\frac{1}{\widetilde{\beta}(\eta)}}$, and the expressions for Ψ_1 and Ψ_1 are given in (2.54) and (2.55). ■

Chapter 3

Willingness to Wait under Risk and Ambiguity: Theory and Experiment

3.1 Introduction

Waiting is an important feature of economic decisions. In many real life situations, individuals can choose between acting immediately or waiting for more favorable conditions. The concept of "option", which finds a wide range of applications in financial economics, is often based on the opportunity (the right) to choose the optimal timing of a pre-specified action.

Among the several factors that affect the value of an option, a prominent role is played by uncertainty. By raising the probability of extreme events, uncertainty makes the possibility to "wait and see" more attractive and increases the option value. Although the academic literature on financial and non financial options has mainly considered uncertainty as risk, it is well known that the distinction between uncertainty with known probabilities, or risk, and uncertainty with unknown probabilities, or ambiguity, has a behavioral significance. In this work we investigate, theoretically and experimentally, the distinct roles of risk and ambiguity on the optimal timing of option exercise.

Optimal timing decisions are not restricted to financial options. On the contrary, they find a wide range of applications. A strand of literature, known under the name of real options theory, applied the option method to model several economic (and non economic) problems.[1] This literature emphasized the role of uncertainty in shaping economic decisions that are, at least to some extent, irreversible. Although in the last two decades the real options approach became a standard method in economics, the empirical evidence for even the most basic predictions of the theory is surprisingly scant.[2] Up to our knowledge, this work is the first attempt to test, in a unifying real options framework, the effects of the two forms of uncertainty, risk and ambiguity, on the optimal timing of option exercise.

As a basis for our experimental study, we need a theoretical framework in which risk and ambiguity are comparable (i.e., risk and ambiguity are related to the same event), which features all the essential elements of the real options theory, and which, at the same time, is easy to implement in a laboratory experiment. The first step of our investigation is to develop a simple real options model that satisfies these requirements.

The basic structure of the model is as follows. A decision maker holds the opportunity to invest in a project by paying a fixed cost. The value of the project grows deterministically over time but, at each instant, the option to invest can disappear at an exogenously-specified expiration rate. If the decision maker invests before expiry, he obtains a payoff equal to

[1] The role of irreversibility in investment problems was originally pointed out by Arrow (1968). Bernanke (1983) shows how irreversibility can explain cyclical movements of investment. The standard formalization of the real option approach for investment decisions is due to MacDonald and Siegel (1986). In more recent years, the real options approach had a wide range of applications. For example, it has been used to study the relation between corporate investment and asset pricing (Berk, Green, and Naik, (1999), Carlson, Fisher, and Giammarino (2004), Aguerrevere (2009)), the timing of mergers and acquisitions (Lambrecht (2004), Morellec and Zhdanov (2005)), innovation investments in competitive markets (Weeds (2002), Huisman and Kort (2003)), debt default (Leland (1994)), and even to model political decisions (Polborn (2006), Keppo et al. (2009)).

[2] Notable exceptions are Guiso and Parigi (1996), Moel and Tufano (2002) and the experimental works of Oprea et al. (2009), Anderson et al. (2010) and List and Haigh (2010). In a recent paper, Kellog (2010) estimates firms' responsiveness to changes in uncertainty using data on oil well drilling in Texas.

the current value of the project minus the investment cost, while he gets nothing otherwise. Thus, there is a value in delaying the investment, but waiting involves an opportunity cost because the future payoff is uncertain. There are two possible states of the world. In the good state, the expiration rate is low, λ_L, while in the bad state, it is high, λ_H. The true value of the expiration rate is unknown at the initial date but the decision maker can learn about the true state of the world. If time progresses and the investment opportunity does not expire, the decision maker can infer that the state of the world is more likely to be good, and he updates his beliefs accordingly.

We distinguish between a risky and an ambiguous scenario. In the scenario that features risk, the decision maker knows the relative probability of the expiration rate being high or low. Risk is given by the spread between high and low expiration rates, for the expected expiration rate being constant. In the investment problem under ambiguity, the decision maker has imprecise information about the probability of the two states of the world. He only knows that this probability lies within a certain interval. We show that risk delays investment, in accord with the real options theory. This result depends on the fact that higher risk raises the upside potential of the option. When the spread between high and low expiration rates is larger, the fact that the option does not expire during a given time interval is a more informative signal. This implies that the decision maker becomes more rapidly confident of the state of the world being good and, therefore, he waits for a higher project value before investing. The effect of ambiguity depends on the decision maker's attitude towards ambiguity. If he is ambiguity averse, investment is undertaken sooner. The reason is that waiting involves an uncertain prospect, while an immediate investment yields a certain payoff. An ambiguity averse decision maker dislikes the uncertainty associated with the waiting region and prefers to invest sooner. In contrast, investment is delayed if the decision maker is ambiguity seeking.

With the support of Figure 3.1, it is useful to interpret our model in comparison with the standard Ellsberg (1961) experiment. In a typical Ellsberg-style setting, subjects decide how much to pay to participate in two lotteries, risky and ambiguous, represented by a draw from an urn with balls of different colors, G and B in the figure (for example, Fox and Tversky (1995), and Halevy (2007)). In the risky lottery the composition of the urn, that is the probability of extracting a B ball, is known and risk is increased by a mean preserving spread of the expected payoff. In the ambiguous lottery the urn composition is (at least partially) unknown. This means that the probability of extracting a B ball has not a unique value, but is defined by a range. In this setting, decision makers disclose their preferences by revealing their *willingness to pay* to participate in the lottery, and a lower willingness to pay for the ambiguous lottery reveals ambiguity aversion.

Our setup is designed to closely resemble Ellsberg's two-urn environment. In the model, the true expiration rate is determined at the initial date by a draw from a distribution which is known in the risky case but unknown in the ambiguous case. Risk is measured by the spread between λ_H and λ_L, while ambiguity is measured by the probability interval for the good and bad state. Instead of revealing their *willingness to pay* as in the standard Ellsberg setup, decision makers disclose their preferences by revealing their *willingness to wait*. A lower willingness to wait in the ambiguous scenario is a sign of ambiguity aversion.

To test the predictions of the model, we replicate it in a laboratory experiment. We initially run three treatments. The first treatment, called *Benchmark*, is a risky treatment in which subjects know the values of the high and low expiration rates and the relative probability of the two states of the world, set equal to 50%. The second treatment, called *Risk*, is also a risky treatment but risk i.e., the spread between the high and low expiration rates, is increased compared to *Benchmark*. In the third treatment, called *Ambiguity*, the values for the high and low expiration rates are as in *Benchmark* but we introduce ambiguity by not giving any information about the relative probability of the two states of the world.

Experimental data strongly support the theoretical prediction about risk. In the treatment *Risk*, the investment decision is delayed compared to *Benchmark*. Somewhat surprisingly, we also find that the investment decision in *Ambiguity* is delayed compared to *Benchmark*. According to the predictions of our theoretical model, this is a sign of ambiguity seeking. To test the robustness of the latter result, we run another treatment,

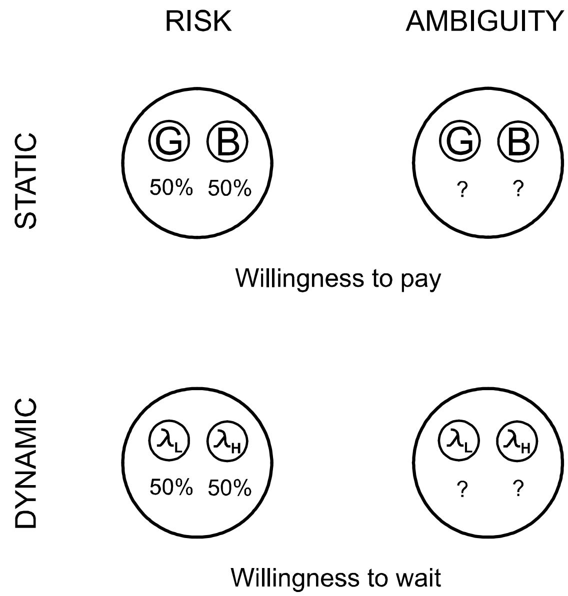

Figure 3.1: Risk and ambiguity in a static and a dynamic setup.

called *Mild Ambiguity* in which we depart from the common practice to provide no information about the probability distribution of the unknown scenario. In this new treatment, growth and expiration rates as in *Benchmark* and *Ambiguity* but subjects have a partial information about the relative probability of the states of the world. Specifically, they know that the probability of the expiration rate being high lies somewhere in between 20% and 80%. Data reveal that in *Mild Ambiguity* investment is still delayed compared to *Benchmark*, though the effect is substantially weaker than in *Ambiguity*. Overall, we find a weak confirmation of an ambiguity seeking attitude.

3.1.1 Related literature

Some recent papers test the real options theory in laboratory experiments. Oprea et al. (2009) take the standard real options model based on geometric Brownian motion and study whether subjects can learn the optimal investment rule. They find supportive evidence that, by experience, individual behavior converges towards optimality. Anderson et al. (2010) study a pre-emption investment game and find that, for the most part, the predictions of the theory are confirmed. List and Haigh (2010) focus on another facet of the theory of investment under uncertainty, the bad news principle, and conclude that experimental data support it.[3]

Theoretical contributions that introduce ambiguity into models of investment under uncertainty are Nishimura and Ozaki (2007) and Miao and Wang (2011). Nishimura and Ozaki (2007) rely on the assumption of ambiguity-aversion and show that increased ambiguity delays investment in projects that generate an infinite flow of ambiguous cash flows. In contrast, our model considers the case where investment yields a certain (unambiguous) payoff and it is therefore closer to the job-search model of Nishimura and Ozaki (2004). Consistent with our result, Nishimura and Ozaki (2004) show that an increase in ambiguity decreases the reservation wage and induce the ambiguity-averse worker to stop the job search earlier. Miao and Wang (2011) further clarify that the sign of the effect of ambiguity depends on whether uncertainty is resolved or not at the time of the investment. When ambiguity affects only the waiting region and the payoff is certain, ambiguity accelerates investment, if the decision maker is ambiguity averse. In contrast, if the final payoff is also ambiguous, investment is delayed. Since we consider the case where the payoff from investment is certain, our model is consistent with the predictions of Miao and Wang (2011).

The investment models of Nishimura and Ozaki (2007) and Miao and Wang (2011) assume that the degree of ambiguity is not reduced by observational data.[4] This work takes a different perspective and considers an environment in which information on the nature of uncertainty is progressively revealed to the decision maker. This choice is dictated by our belief that the combined presence of learning and ambiguity is an accurate description of many dynamic real-life situations. Decision makers often face situations that can be described as ambiguous. However, if the environment remains stable for a certain amount of time, they have the

[3] All these experiments exclusively focus on investment under risk.

[4] Miao and Wang (2011) present a job-matching model with learning.

possibility to learn the nature of uncertainty. Hence, we think of an economic environment where the emergence of new (ambiguous) scenarios is followed by intervals of relative stability, which allow individuals to learn.

The remainder of this work is organized as follows. Section 3.2 develops a model of investment under risk and ambiguity. Section 3.3 describes the experimental design and the testable hypotheses. Section 3.4 presents the empirical results, and Section 3.5 concludes.

3.2 The Model

3.2.1 A simple stopping problem

We first present a simple optimal stopping problem that will serve as a building block for our analysis. Time is continuous and labeled by $t \in [0, \infty)$. A risk neutral decision maker (DM henceforth) discounts the future at rate r and has an opportunity (option) to invest in a project of value V_t by paying a fixed cost equal to C.[5] The value of the project grows deterministically over time and, therefore, the DM has an incentive to wait. However, the opportunity to invest can expire and disappear at a random time. This means that if the DM invests at time t before the opportunity expires, he obtains a payoff $V_t - C$, while he gets nothing otherwise. The DM has to decide when to invest.

The value of the project V_t evolves according to

$$dV_t = \mu V_t dt, \tag{3.1}$$

where $\mu > 0$ is the growth rate. It follows that the value of the project at time t is equal to

$$V_t = V_0 e^{\mu t}. \tag{3.2}$$

At each instant, the investment opportunity may vanish with a strictly positive probability. The expiration of the investment option is modeled as a Poisson shock with mean arrival rate $\lambda > 0$. This means that, over a period of time Δt, the DM loses the opportunity to invest with probability

[5] The assumption of risk neutrality does not affect the qualitative predictions of the model, which are the object of our experimental analysis. In Appendix 3.A.6, we show that the fundamental results hold even when the decision maker is risk averse.

$\lambda \Delta t$. We assume that $\lambda + r > \mu$ to guarantee that the option to invest will be exercised in finite time.

We denote by V_K^* the project value such that, if $V_t \geq V_K^*$, it is optimal for the DM to exercise the investment option. The subscript K stands for "*known*," to indicate that the DM has a perfect knowledge of the expiration rate λ. The following proposition provides the expression for the optimal investment trigger.

Proposition 3.1 *The optimal investment trigger is*

$$V_K^* = \frac{\beta}{\beta - 1} C, \tag{3.3}$$

where $\beta = (\lambda + r) / \mu > 1$*.*

The ratio $\beta / (\beta - 1) > 1$ gives the proportion by which the value of the project should grow above the cost to induce the DM to invest. A higher λ decreases the investment trigger. Intuitively, if the probability of losing the investment opportunity is larger, the DM will exercise the investment option sooner. In contrast, when μ is higher, he will postpone the investment in order to exploit the larger growth potential.

3.2.2 Unknown expiration rate

Consider, now, a slightly modified setting. As in the previous section, the DM has the opportunity to invest in a project of value V_t which grows according to (3.1). At each instant, the option to invest expires with a strictly positive probability and the expiration rate is determined by the intensity of a Poisson process. However, the expiration rate is unknown. The DM knows that two states of the world are possible. In the bad state, the expiration rate, λ_H, is high. In the good state the expiration rate, λ_L, is low ($\lambda_H > \lambda_L$ holds). The DM knows the values of the two λs but does not know the realization of the state. To ensure that the investment problem has always a finite solution, we assume that

$$r + \lambda_L > \mu. \tag{3.4}$$

Suppose that the DM has a subjective belief about the relative probability of the two states. Specifically, he thinks that the intensity of the expiration rate is λ_L with probability $p \in (0,1)$. For the moment, we do not specify how this belief is formed at the initial time $t = 0$. However, we do specify how it evolves over time. If at $t = 0$ the DM finds it optimal not to invest immediately, he waits for larger values of the project. By waiting, the DM observes the investment payoff to rise according (3.1) and the (non)occurrence of the expiry. If the DM waits and the option to invest does not expire, he updates his belief in a Bayesian fashion. According to Bayes' rule, after an interval Δt, DM's posterior belief is:

$$p_t + \Delta p_t = \frac{p_t \left(1 - \lambda_L \Delta t\right)}{p_t \left(1 - \lambda_L \Delta t\right) + \left(1 - p_t\right) \left(1 - \lambda_H \Delta t\right)}. \tag{3.5}$$

Taking the limit for $\Delta t \to 0$ and rearranging yields the instantaneous change in belief:

$$dp_t = p_t \left(1 - p_t\right) \left(\lambda_H - \lambda_L\right) dt. \tag{3.6}$$

Equation (3.6) can be interpreted as the speed at which the DM learns about the true state of the world. The speed of learning is proportional to the difference $\Delta\lambda = \lambda_H - \lambda_L$. The explanation is that, when the difference between λ_H and λ_L is large, the fact that during a given time interval the option to invest does not vanish is very informative. Then, the DM becomes rapidly confident that the true expiration rate is low and, therefore, p_t increases quickly. Equation (3.6) implies that:

$$p_t = \frac{p_0 e^{\lambda_H t}}{\left(1 - p_0\right) e^{\lambda_L t} + p_0 e^{\lambda_H t}}, \tag{3.7}$$

where p_0 is the belief at time $t = 0$.

In the remainder, we distinguish two different scenarios. First, we have a risky scenario ("Risk"), in which the DM knows the relative probability of the two states of the world at the initial time. Second, we consider and an ambiguous scenario ("Ambiguity"), in which the probability is unknown.

3.2.3 Risk

In the scenario that we call "Risk," the DM has a single initial prior, denoted by $\overline{p}$, which defines the probability of the expiration rate being λ_L at time 0. The analysis here is consistent with the assumption that the

DM knows exactly the true probability with which the state of the world is selected, or that he can form a single subjective prior that represents his beliefs. In either case the standard expected utility model can be applied. For a given initial expected expiration rate, we measure risk as the spread between λ_H and λ_L for the expected expiration rate at the initial time being constant. In other words, risk is measured by the difference $\Delta\lambda = \lambda_H - \lambda_L$ for a given $\overline{p}\lambda_L + (1-\overline{p})\,\lambda_H$. The goal of this section is to study the effect of risk on the timing of investment. We show that, in accord with the standard result from real options theory, risk delays investment.

The decision problem is analogous to the one described in Section 3.2.1. The DM sees the value V_t growing deterministically over time and has to decide when to invest. Appendix 3.A.2 proves that the value of the investment option must satisfy the following Bellman equation:

$$F_t = \max\left\{V_t - C,\ F_t + dF_t - \left[r + (1-p_t)\,\lambda_H + p_t\lambda_L\right] F_t dt\right\}. \tag{3.8}$$

Equation 3.8 shows that, at each instant, the DM has to choose between investing immediately to get the payoff $V_t - C$, or to postpone the investment. Waiting allows capturing the benefits of the positive growth rate, but involves the cost associated with the risk of losing the investment option. The option value F_t depends on two time varying variables, the value of the project V_t and the belief p_t. In Appendix 3.A.2 we also show that, since both V_t and p_t are deterministic functions of time, the dimensionality of the problem can be reduced to only one state. Specifically, we can write DM's belief p_t in terms of V_t, and use the value of the project as the only state. The transformed belief function is denoted by $p(V_t)$, and its explicit expression is given by

$$p(V_t) = \frac{\Omega_R V_t^{\frac{\lambda_H-\lambda_L}{\mu}}}{1+\Omega_R V_t^{\frac{\lambda_H-\lambda_L}{\mu}}}, \tag{3.9}$$

where $\Omega_R = \pi_0 V_0^{\frac{\lambda_L-\lambda_H}{\mu}}$.

We denote the investment trigger by V_R^*. If $V_t \geq V_R^*$, it is optimal for the DM to invest, while it is optimal to wait otherwise. The following proposition holds.

Proposition 3.2 *The investment trigger V_R^* is implicitly defined by the condition*

$$V_R^* = \frac{\beta_R (V_R^*)}{\beta_R (V_R^*) - 1} C, \tag{3.10}$$

where $\beta_R (V_t) = [r + p (V_t) \lambda_L + (1 - p (V_t)) \lambda_H] / \mu$.

Then, we can study the qualitative effects of an increase in risk.

Proposition 3.3 *The investment trigger V_R^* is non-decreasing in risk.*

Proposition 3.3 says that, as our measure of risk increases, investment is in general delayed. The explanation relies on the belief updating mechanism. The proportion by which the project value V_t should grow above the investment cost to induce the DM to invest is determined by $\beta_R (V_t)$, which increases with the expected expiration rate $p (V_t) \lambda_L + (1 - p (V_t)) \lambda_H$. According to equation (3.6), the posterior belief $p (V_t)$ grows faster when the spread between λ_H and λ_L is larger. This implies that, other things being equal, at each time $t > 0$ the expected expiration rate, and therefore $\beta_R (V_t)$, will be lower in a riskier scenario (i.e. when the spread $\Delta\lambda$ is larger). Since $\beta_R (V_t) / (\beta_R (V_t) - 1)$ increases when $\beta_R (V_t)$ decreases, it follows that equation (3.10), which determines the investment trigger, will be satisfied for a larger value of V_t. To have a more intuitive explanation of the mechanism at work, one could interpret the result of Proposition 3.3 as follows. When risk is higher, learning proceeds faster and the DM becomes more rapidly confident that the state of world is the good one (i.e. the expiration rate is low). For this reason, he finds it optimal to longer exploit the benefits of the positive drift and to exercise the option at a higher project value.[6]

[6] In the standard real options investment problem, risk is measured by the volatility coefficient of the stochastic process (typically a Geometric Brownian Motion) which defines the value of the underlying. Higher risk increases the value of waiting because it enables the option holder to realize a larger upside potential avoiding the downside risk. Our measure of risk, although via a different mechanism, preserves this intuition. A mean preserving spread between λ_H and λ_L at the initial date increases the likelihood of a bad outcome (i.e. the option expiry) in case the high expiration rate is drawn. But at the same time, via the effect of the learning mechanism, it also increases the upside potential of the option. When risk increases, the DM becomes more rapidly confident that the state of the world is the good one and, for this reason, he considers more likely the possibility that V_t will reach larger values.

The delay of investment in response to a higher risk depends on a real options effect (risk makes the option to wait more valuable) and it is independent of DM's risk attitude. In fact, in Appendix 3.A.6 we solve the model for a risk-averse DM with CRRA utility and show that, although risk aversion affects the option exercise strategy, the result of Proposition 3.3 still holds. In particular, we show that introducing risk aversion is equivalent to reducing the proportional growth rate μ and has the effect to accelerate the exercise of the investment option. However, for a given degree of risk aversion, a higher risk increases the upside potential of the option and delays investment,

3.2.4 Ambiguity

In the scenario that we call "Ambiguity" the DM has only an imprecise knowledge about the initial probability of the two states of the world. For this reason, the DM cannot form a single prior but can only identify a set of plausible beliefs. Let $\mathcal{P}_t$ be a closed compact set of plausible beliefs. At the initial time $t = 0$, the initial probability set is defined by $\mathcal{P}_0 = [\overline{p} - \varepsilon, \overline{p} + \varepsilon]$. Here, $\overline{p}$ simply denotes the middle point of the set of plausible initial probabilities while $\varepsilon \in (0, \min[\overline{p}, 1 - \overline{p}])$ is a measure of the initial degree of ambiguity. The goal of this section is to study how ambiguity affects the timing of investment. We prove that the effect of ambiguity depends on the attitude of the DM. Spefically, ambiguity accelerates investment if the DM is ambiguity averse but delays investment if the DM is ambiguity seeking.

As in the risky case described in Section 3.2.3, the DM learns about the true state of the world. If time progresses and the investment option does not vanish, he becomes progressively more confident that the true expiration rate is λ_L. However, contrary to the risky case, the learning process does not involve a single prior but the entire set of plausible beliefs $\mathcal{P}_t$. To capture the learning process, we assume that the DM updates $\mathcal{P}_t$ prior-by-prior. This means that each belief in $\mathcal{P}_t$ evolves according to the dynamic described in (3.6) and, at a generic time t, satisfies (3.7).[7]

[7] Prior-by-prior Bayesian updating, is a common rule to update ambiguous beliefs. It was proposed, among others, by Wasserman and Kadane (1990) and Jaffray

Because beliefs remain in the same order, the posteriors originating from $\overline{p}-\varepsilon$ and $\overline{p}+\varepsilon$ represent the worst and best case beliefs, and define the boundaries of $\mathcal{P}_t$. Call p_t^+ the posterior belief under the best case scenario, and p_t^- the posterior belief under the worst case. At time t, those two beliefs are equal to (cf. equation (3.7)):

$$p_t^+ = \frac{(\overline{p}+\varepsilon)\, e^{\lambda_H t}}{(1-\overline{p}-\varepsilon)\, e^{\lambda_L t} + (\overline{p}+\varepsilon)\, e^{\lambda_H t}}, \tag{3.11}$$

and

$$p_t^- = \frac{(\overline{p}-\varepsilon)\, e^{\lambda_H t}}{(1-\overline{p}+\varepsilon)\, e^{\lambda_L t} + (\overline{p}-\varepsilon)\, e^{\lambda_H t}}, \tag{3.12}$$

while the plausible set of beliefs, which defines the range of ambiguity, is given by $\mathcal{P}_t = \left[p_t^-, p_t^+\right]$. It can be easily checked that the difference between p_t^+ and p_t^- decreases with time as both beliefs asymptotically converge to one. In other words, the range of ambiguity shrinks over time.

We do not restrict a priori DM's attitude towards ambiguity. To capture both ambiguity aversion and ambiguity seeking in a simple fashion, we adopt the α-MEU model proposed by Ghirardato et al. (2004). This model is a combination of *maxmin* and *maxmax* expected utility, where an agent's utility (in our case the investment option value) is a fraction α of the minimum plus a fraction $1-\alpha$ of the maximum expected utility, over the feasible set of priors. A larger α implies a relatively higher degree of ambiguity-aversion. When $\alpha = 1$, the agent considers only the worst-case among all possible outcomes and the model reduces to the *maxmin* model of Gilboa and Schmeidler (1989). When $\alpha = 0$, the agent considers only the best-case. In our setting, the α-MEU speification implies that the investment opportunity is a convex combination of the investment option evaluated under the worst case and best case beliefs, p_t^- and p_t^+.

As shown in Appendix 3.A.4, the investment option satisfies the following Bellman equation:

$$F_t = \max\left\{V_t - C,\ F_t + dF_t - \left[\alpha R_t^- + (1-\alpha)\, R_t^+\right] F_t dt\right\}, \tag{3.13}$$

where

$$R_t^- = r + \left(1-p_t^-\right)\lambda_H + p_t^- \lambda_L \text{ and } R_t^+ = r + \left(1-p_t^+\right)\lambda_H + p_t^+ \lambda_L. \tag{3.14}$$

(1992) and axiomatized in Pires (2002). Gilboa and Schmeidler (1993) provide a non-exhaustive presentation of alternative rules to update ambiguous beliefs.

Employing the same transformation as in Section 3.2.3, posterior beliefs p_t^- and p_t^+ can be written as functions of the project value V_t. Denote by $p^-(V)$ and $p^+(V)$ the worst and best-case transformed beliefs, respectively, and define the "weighted belief" as $p_A(V) = \alpha p^-(V) + (1-\alpha)p^+(V)$. The expressions for $p^-(V)$ and $p^+(V)$ are given by

$$p^-(V_t) = \frac{\Omega^- V_t^{\frac{\lambda_H-\lambda_L}{\mu}}}{1+\Omega^- V_t^{\frac{\lambda_H-\lambda_L}{\mu}}} \text{ and } p^+(V_t) = \frac{\Omega^+ V_t^{\frac{\lambda_H-\lambda_L}{\mu}}}{1+\Omega^+ V_t^{\frac{\lambda_H-\lambda_L}{\mu}}}, \tag{3.15}$$

where $\Omega^- = \frac{\overline{p}-\varepsilon}{1-(\overline{p}-\varepsilon)} V_0^{\frac{\lambda_L-\lambda_H}{\mu}}$ and $\Omega^+ = \frac{\overline{p}+\varepsilon}{1-(\overline{p}+\varepsilon)} V_0^{\frac{\lambda_L-\lambda_H}{\mu}}$.

Proposition 3.4 *The investment trigger V_A^* is implicitly defined by the condition*

$$V_A^* = \frac{\beta_A(V_A^*)}{\beta_A(V_A^*)-1} C, \tag{3.16}$$

where $\beta_A(V_t) = [r + p_A(V_t)\lambda_L + (1-p_A(V_t))\lambda_H]/\mu$.[8]

To study the effect of ambiguity on the timing of investment, we first define the concept of ambiguity neutrality. We call ambiguity neutral a DM whose investment strategy in an ambiguous scenario does not differ to his strategy in the unambiguous scenario, other things being equal. More formally, let the investment trigger momentarily be dependent on the degree of ambiguity, i.e., $V_A^*(\varepsilon)$. A DM is ambiguity neutral if he is characterized by an ambiguity attitude parameter $\alpha = \alpha^*$ such that $V_A^*(0) = V_A^*(\widetilde{\varepsilon})$, where $\widetilde{\varepsilon}$ is a given degree of ambiguity. In fact, since the unambiguous scenario $\varepsilon = 0$ corresponds to the risky case outlined in the previous section, ambiguity neutrality can equivalently be defined by the condition $V_R^* = V_A^*$, other things being equal. Using (3.10) and (3.16), and solving the previous equality with respect to α yields:

$$\alpha^* = \mu \frac{\beta(V_t) + (\beta_R(V^*)-1)/\left(\beta_R^-(V^*)-1\right)}{(\lambda_H-\lambda_L)(p^+(V^*)-p^-(V^*))},$$

[8] When $\overline{p} = 0.5$ and the degree of ambiguity is maximal ($\varepsilon = 0.5$) worst and best case beliefs are $p^- = 0$ and $p^+ = 1$ and are not subject to updating dynamics (cf. 3.6). In this special case, the investment trigger has explicit solution $V_A^* = \overline{\beta}_A / \left(\overline{\beta}_A - 1\right) C$, where $\overline{\beta}_A = [r + (1-\alpha)\lambda_L + \alpha\lambda_H]/\mu$. Assumption (3.4) ensures that $\overline{\beta}_A > 1$.

where $\beta_R^-(V^*) = [r + p^-(V^*)\lambda_L + (1 - p^-(V^*))\lambda_H]/\mu$ and $V^* = V_R^* = V_A^*$. It is worth noting that the parameter α^* is a function of ε.[9] This means that the value of α which identifies ambiguity neutrality is not unique but varies in relation to the degree of ambiguity. In other words, an ambiguity neutral DM will be identified by a different α depending on the degree of ambiguity.

Ambiguity aversion and ambiguity seeking are defined as follows. We call ambiguity averse a DM that weighs relatively more the worst case scenario compared to the ambiguity neutral DM. That is, a DM is ambiguity averse if $\alpha > \alpha^*$. Similarly, a DM is ambiguity is seeking if $\alpha < \alpha^*$. The following proposition shows the effect of ambiguity on the investment decision.

Proposition 3.5 *Compared to the unambiguous case, $\varepsilon = 0$, ambiguity accelerates investment if the DM is ambiguity averse, while it delays investment if the DM is ambiguity seeking.*

Proposition 3.5 reveals that the effect of ambiguity on the investment decision crucially depends on the attitude towards ambiguity. If the DM is ambiguity averse, investment occurs earlier compared to the unambiguous case, while the opposite holds if the DM is ambiguity seeking. The explanation relies on the nature of the payoff structure. While waiting involves an uncertain prospect, because the option can vanish at each instant, investment yields a certain payoff. When the DM is ambiguity-averse, he dislikes the uncertainty associated with the waiting region and prefers to invest sooner. In contrast, an ambiguity seeking DM will wait longer to obtain a larger payoff. As shown in Appendix 3.A.6, the result of Proposition 3.5 does not depend on the risk attitude of the DM.

3.3 Experimental design and testable hypotheses

While in the theoretical model time is continuous, in the laboratory implementation time must proceed at discrete steps. We approximate continu-

[9] Indeed ε enters the expressions for $p^+(V^*)$ and $p^-(V^*)$.

	High exp. rate	**Low exp. rate**	**1−$\overline{p}$**
Benchmark	10%	5%	50%
Risk	11%	4%	50%
Ambiguity	10%	5%	no info

Table 3.1: **Parameterization for the treatments *Benchmark*, *Risk* and *Ambiguity*.**

ous time by setting the time interval equal to 0.1 seconds. In each interval, the project value grows and expires according to the chosen growth and expiration rates. To convey a clear information to the subjects, we communicate the growth and expiration rates *per second*.

To test the predictions of the model, we run three treatments named *Benchmark*, *Risk*, and *Ambiguity*. In all treatments, the investment cost C is set equal to 10 Euros, the initial value V_0 is set equal to 9.8 Euros, while the growth rate is equal to $\mu = 0.0036$ every 0.1 seconds. This is corresponds to a growth rate equal to 3% per second.[10] In the treatment *Benchmark* the initial probability that the expiration rate is low is known and equal to $\overline{p} = 0.5$. The high expiration rate is set equal to $\lambda_H = 0.0105$, which means 10% per second, while the low expiration rate is $\lambda_L = 0.0051$, that is 5% per second. This implies that the initial expected expiration rate is equal to 7.5% per second. In the treatment *Risk* we test the effects of an increase in risk. We leave the initial probability $\overline{p}$ and the growth rate μ unaffected. Then, we set $\lambda_H = 0.0116$, i.e., 11% per second, and $\lambda_L = 0.0041$, i.e., 4% per second. Therefore, the spread between high and low expiration rates has increased, while the initial expected expiration rate is still equal to 7.5% per second. In the treatment *Ambiguity*, we test the effects of ambiguity. We set λ_H and λ_L as in *Benchmark* but do not give any information about the initial relative probability of the expiration rate being high or low.[11] A summary of the parameterization for the three treatments is found in Table 3.1.

The choice of parameter values is driven by a number of considera-

[10] If an event occurs with probability x every 0.1 seconds, it occurs with probability $y = 1 - (1 - x)^{10}$ every second.

[11] In practice the probability that the true expiration was low in each period was 50% as in *Benchmark*. But this information was not communicated to the subjects.

tions. We set the relative probability of the two states of the world equal to 50% in the risky treatments *Benchmark* and *Risk* while we do not give any information in the ambiguous treatment *Ambiguity*. This choice is to conform to a standard version of the Ellsberg experiment, in which good and bad outcomes in the risky urn have equal probabilities while subjects are told nothing about the distribution of the unknown urn. Although the task is relatively simple, subjects have to wait and just click a button when they decide to invest, the time interval and expiration rates should set a challenging but at the same time "comfortable" environment. For example, the time interval should give the feeling of continuous time but it should not be "too short". Similarly, expiration rates cannot be set too high to give subjects sufficient room to wait, and the spread between high and low expiration rates should be appreciable but not too large, to avoid that the task of distinguishing between states of the world becomes trivial. Furthermore, under the chosen parameterization, the three treatments should give sufficiently distinguishable theoretical predictions for the investment trigger. For our parameter values the predicted investment trigger is 19.32 in *Benchmark*, and 30.53 in *Risk*. Notice how a relatively small increase is risk (from 10% and 5% to 11% and 4% for the high and low expiration rates) generates a sharp rise in the predicted investment trigger. In *Ambiguity* the investment trigger depends on the parameter α which defines DM's attitude towards ambiguity, and a comparison with the predicted trigger *Benchmark* is not readily available. To see whether *Ambiguity* and *Benchmark* give sufficiently distinguishable predictions we adopt the following strategy. We first find the value of α which defines risk neutrality. For our parameterization this value is equal to $\alpha = 0.245$. Then, assuming that subjects are ambiguity averse, we consider the investment trigger of a DM with a mild degree of ambiguity aversion, i.e., $\alpha = 0.5$. For this value of α, the predicted investment trigger is 16.67, which is sufficiently lower than the trigger in *Benchmark*.

From Propositions 3.3 and 3.5 the next two experimental hypotheses follow.

Hypothesis 1 *The investment trigger in Risk is larger than the one in Benchmark.*

Figure 3.2: Initial screen information.

Hypothesis 2 *If subjects are ambiguity averse, the investment trigger in Ambiguity is lower than the one in Benchmark. If subjects are ambiguity seeking, the investment trigger in Ambiguity is larger than the one in Benchmark.*

Hypothesis 1 states that an increase in risk delays investment. If subjects rationally update their beliefs, the learning mechanism outlined in Section 3.2.3 should induce them to wait longer before investing. Hypothesis 2 comes from the fact that, as shown in Proposition 3.5, the presence of ambiguity should lead to an early exercise of the investment option if subjects are ambiguity averse, and the other way around.

3.3.1 Procedures

Subjects played the same investment game for 30 rounds. At the beginning of each round the computer screen shows the parameter values. In the treatments *Benchmark* and *Risk*, the screen displays the value for λ_H, λ_L

Figure 3.3: Screen display during the treatment.

and the probability that the expiration rate is high (Figure 3.2). In the treatment *Ambiguity* the screen displays only the values for λ_H and λ_L, without any information about the probability of the two states of the world. By clicking the button "OK" located at the bottom-right of the screen, the DM starts the experiment and a new screen with the values for V_t and the cost C appears (Figure 3.3). Subjects see the project value V_t growing according to (3.1) and decide when to exercise the investment option by clicking the button "INVEST". Upon investing, they obtain a payoff $V_t - C$.

The experiment was conducted at CentERLab at Tilburg University and the experimental subjects were 55 students of Tilburg University recruited by an on-line recruitment software. Since subjects played the investment game for 30 rounds, we have a total of 1650 observations. Participation was voluntary and no subject participated in more that one treatment. Groups of 18 subjects participated in the treatments *Benchmark* and *Risk*, while 19 subjects participated in the treatment *Ambiguity*.

Subjects were paid 5 Euros as a showup fee. The experiment was programmed and conducted with the software z-Tree (Fischbacher (2007)).

Before starting the investment game, each subject played a minimum of 20 practice rounds. In the practice rounds, subjects played the game described in Section 3.2.1 in which the expiration rate is known. In the treatments *Benchmark* and *Ambiguity* subjects played 10 practice rounds with an expected expiration rate equal to 10% and 10 rounds with an expected expiration rate equal to 5%. In the treatment *Risk* subjects played 10 practice rounds with an expected expiration rate equal to 11% and 10 rounds with an expected expiration rate equal to 4%. By raising their hands, subjects could call the experimenter(s) and ask for more practice rounds.

At the beginning of each treatment instructions were read aloud. Subjects were seated in isolated computer terminals. Earnings were paid at the end of the experimental sessions. To avoid wealth effects, subjects were paid for only one of the 30 rounds. The payment round was chosen at random at the end of the experiment. The average earning was 9.10 Euro, including the showup fee. Sessions lasted about fifty minutes, including reading of instructions and payment.

3.4 Results

As mentioned in Section 3.3, we set the initial project value below the cost of investment ($C = 10$ and $V_0 = 9.8$). In some cases the investment opportunity expired when $V_t < C$. There were three of these "early" expirations in *Benchmark*, one in *Ambiguity*, and none in *Risk*. Since it is not rational to invest when $V_t < C$ (and the computer program forbids it), early expirations do not convey any information about subjects' willingness to wait. For this reason, they are dropped from our dataset.

The remaining data are right-censored. A number of investment decisions are not observed because the option expires before subjects invest. In the treatment, *Benchmark* there were 244 cases out of 537 (45% of the total) in which the option expired before subjects decided to invest. Expirations were 278 out of 540 (51% of the total) in *Risk* and 307 out of

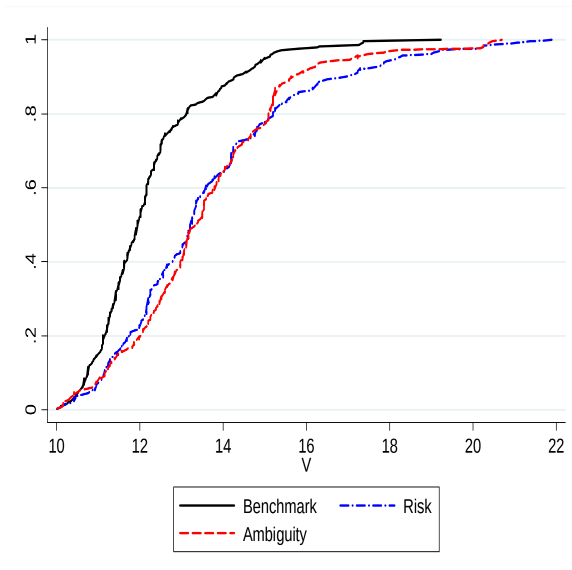

Figure 3.4: Empirical CDFs of the observed investment trigger for the treatments *Benchmark* (solid line), *Risk* (dash-dotted) and *Ambiguity* (dashed).

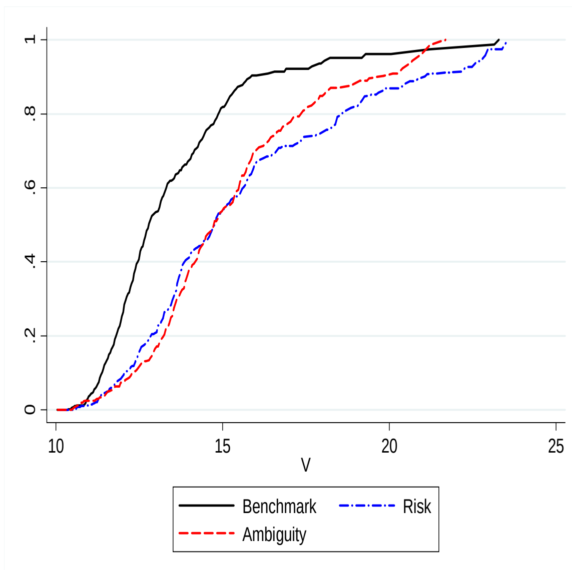

Figure 3.5: Product-Limit estimate of CDFs of the investment trigger for the treatments *Benchmark* (solid line), *Risk* (dash-dotted) and *Ambiguity* (dashed).

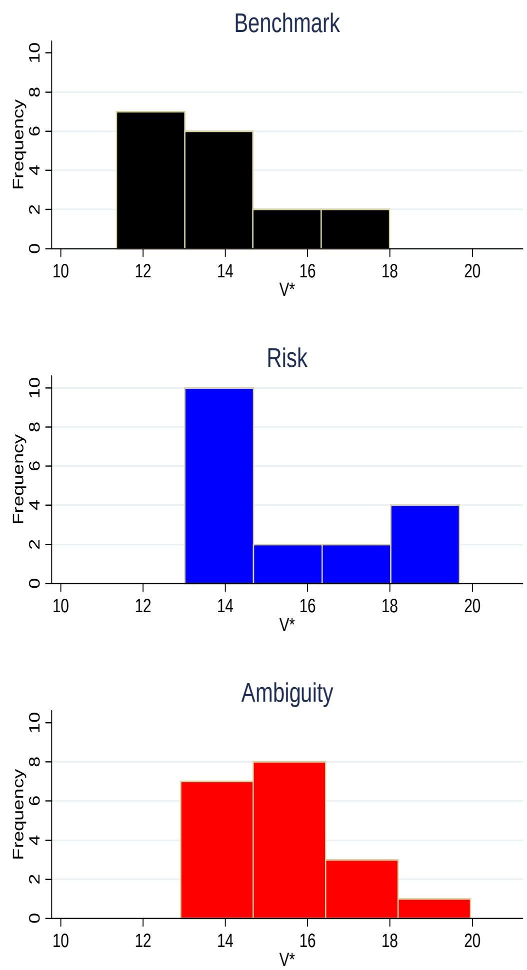

Figure 3.6: Product-Limit mean estimate of the investment trigger for the treatments *Benchmark*, *Risk* and *Ambiguity*.

	Benchmark	**Risk**	**Ambiguity**
		Mean ± Std.Err	
Pooled PL	13.59 ± 0.15	15.68 ± 0.20	15.29 ± 0.15
By-subject PL	13.61 ± 0.42	15.48 ± 0.53	15.14 ± 0.39

Table 3.2: Product-Limit estimate of the mean of the investment exercise trigger. The Pooled PL row shows the estimate assuming i.i.d. observations. The by-subject PL row shows the Product-Limit estimated mean across individual subjects.

569 (54% of the total) in *Ambiguity*.

The analysis of the data follows the same lines as Oprea et al. (2009). As a preliminary step, we drop the censored observations, i.e. the cases where the option expired before subjects decided to invest, and look at the investment pattern for the subsample of observed investment decisions. Figure 3.4 plots the cumulative density functions (CDFs) of the empirically observed exercise trigger. For each value of the project, the figure shows the proportion of subjects who exercised the investment option. A shift of the curve to the right means that, for a given value of V_t, fewer people exercised the option and therefore implies a stronger willingness to wait. A visual inspection of the figure immediately reveals a clear pattern. The CDF for the treatment *Risk* lies to the right to the one of *Benchmark*, which is consistent with the prediction of Hypothesis 1. Also, the figure shows that the CDF for the treatment *Ambiguity* is shifted to the right compared to the one of *Benchmark*. According to Hypothesis 2, this is consistent with ambiguity seeking.

The analysis conducted on the restricted sample of empirically observed investment decisions may potentially convey a misleading picture. Censored observations are also informative because subjects voluntarily decided to wait for the project value to grow (at least) until the moment in which the expiration occurred. This means that the restricted sample of uncensored observations suffers from a downward bias. To deal with the problem of censored data we use the Product-Limit estimator (Kaplan and Meier (1958)), which provides a non-parametric method to estimate the CDFs while accounting for random censoring.

In including the right censored observations, we first assume that observations are i.i.d. and estimate the CDFs using pooled data across subjects. Then, we formally compare the pooled Product-Limit estimates for the CDFs using the log-rank test. Product-Limit estimates of the CDFs for the three treatments are reported in Figure 3.5. The figure confirms the intuition suggested by the restricted sample of uncensored observations. Both risk and ambiguity delay investment. We use the log-rank test to verify the null hypothesis of equality between CDFs. A pair wise test rejects the null hypothesis of equality between the treatments *Risk* and *Benchmark*, and the treatments *Ambiguity* and *Benchmark* at 1% level of significance ($p = 0.000$).

The analysis on pooled data hinges upon the i.i.d. assumption, which implies that investment decisions are uncorrelated across subjects. If this is not the case, and different subjects behave in a different way, the i.i.d. assumption is violated and standard errors are likely to be underestimated. To account for within subject dependence, we construct for each individual a Product-Limit estimate for the average of the investment trigger. Figure 3.6 shows histograms of the by-subject means for each of the three treatments. The figure reveals that, on average, subjects in the treatments *Risk* and *Ambiguity* invest later than in *Benchmark*. Table 3.2 reports means and standard errors of the investment trigger for the pooled and by-subject data. Means in *Risk* and *Ambiguity* are larger than the estimated mean in *Benchmark*. We apply a pair wise Mann–Whitney test to compare sample means for the by-subject estimates. The null hypothesis of equality between *Benchmark* and *Risk* and between *Benchmark* and *Ambiguity* is rejected at 1% level of significance ($p = 0.004$ and $p = 0.007$ respectively). Thus, the fundamental intuition of the analysis with pooled data is confirmed.

The effect of risk is consistent with the theory and supports the logic behind the belief updating mechanism outlined in Section 3.2.2. When the spread between high and low expiration rates widens, subjects become more rapidly confident that the true expiration rate is low and they consistently delay the exercise of the investment option. The effect of ambiguity is striking at least on two different grounds. First of all, ambiguity does affect investment decisions. Since we adopt a between-subject design, this is far from being an obvious result. As showed in Fox and Tversky (1995), ambiguity is more likely to affect individual decisions when subjects compare choices between ambiguous and non-ambiguous environments (within-subjects comparative ignorance), but the effect of ambiguity tends to disappear in between-subject designs. This is not the case in our experiment. Second, the delay of investment in *Ambiguity* compared to *Benchmark* implies that subjects are more willing to face the uncertainty associated with the continuation region in the ambiguous scenario. This is consistent with subjects being ambiguity seeking.

From a quantitative point of view, it is worth noting that subjects exercised the option much earlier than what predicted by the theoretical model, at least for the treatments *Benchmark and Risk* where a comparison is meaningful (the theoretical prediction for the trigger is 19.32 in *Benchmark*, and 30.53 in *Risk*).[12] Figures 3.4 and 3.5 reveal a remarkably strong result. That is, there are practically no cases in which subjects waited for the investment trigger predicted by the model. This result can be easily explained by the fact that, while the model is solved under the assumption of risk neutrality, subjects are risk averse. As shown in Appendix 3.A.6 the effect of risk aversion is indeed to induce an earlier exercise of the investment option. The premature option exercise can also be due a form of loss aversion. Since the investment opportunity can suddenly disappear, subjects may be particularly cautious about waiting to get a larger payoff. Finally, another (not necessarily alternative) explanation can be that, contrary to what assumed in the theoretical model, subjects are not perfectly Bayesian and their belief respond sluggishly to observations. If this happens, the updating dynamic is slower than what defined by (3.6). Then, subjects becomes less rapidly confident that the true state of the world is the good one and tend to invest earlier.

In a different setting, Oprea et al. (2009) also find that in the large majority of the cases, subjects invest prematurely compared to the risk neutral optimal trigger. However, Oprea et al. (2009) also show that, in later rounds, subjects delay the exercise of the option and approach optimality with experience. In contrast, as shown in the next section,

[12] In *Ambiguity* the trigger depends on the ambiguity attitude and we cannot readily compare the theoretical predictions with the empirical results.

	Rounds 1-15	Rounds 16-30
	Mean ± Std.Err	
Benchmark	13.37 ± 15	13.78 ± 0.26
Risk	15.51 ± 0.28	15.84 ± 0.28
Ambiguity	15.45 ± 0.23	15.13 ± 0.21

Table 3.3: Product-Limit estimate of the mean of the investment trigger in rounds 1-15 and 16-30.

in our data there is no evidence of an appreciable change in subjects' investment strategy over time.

3.4.1 Inter-round learning

In the theoretical model the DM holds only one option to invest and the optimal investment rule is defined as a trigger strategy which prescribes that investment should occur whenever V grows above a certain value. In the experimental implementation, for data requirements, the investment game was repeated for 30 rounds. The repetition of the investment game raises the possibility that subjects may adopt different strategies in different rounds, for example because of inter-round learning or experimentation. In our experiment, this concern is mitigated by the fact that subjects play the same game in every round and the initial practice period should help them to elaborate an optimal strategy. Furthermore, since subjects are paid for only one of the 30 rounds, possible wealth effects that could alter their behavior over time are ruled out.

In this section we verify whether subjects' investment strategy displays substantial changes across rounds. To begin with, for each of the three treatments, we use pooled data and split the sample in two sub-periods. The first includes rounds 1 to 15, and the second rounds from 16 to 30. We study whether there exists a difference between earlier and later periods by comparing Product-Limit estimates for the CDF in the two subsamples.[13] Figure 3.7 shows that the estimated CDFs almost perfectly overlap in all treatments, suggesting that there is no appreciable change in behavior

[13]Comparing the investment strategies in sub-periods 1-10 and 21-30 yields the same results.

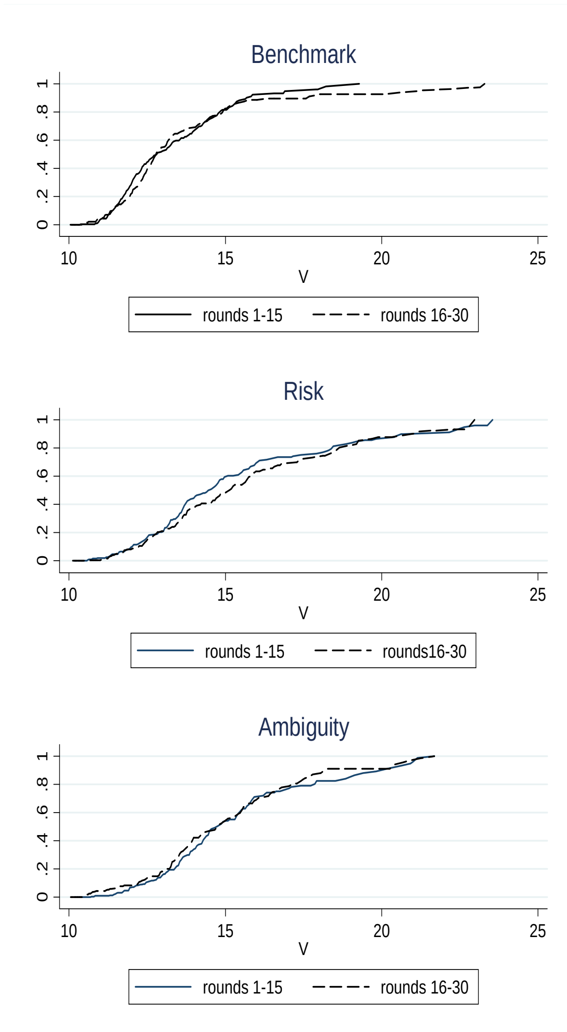

Figure 3.7: Product-Limit estimate of CDFs of the investment trigger for the treatments *Benchmark*, *Risk* and *Ambiguity*, splitting the sample in two subperiods, rounds 1-15 (solid line) and 16-30 (dashed).

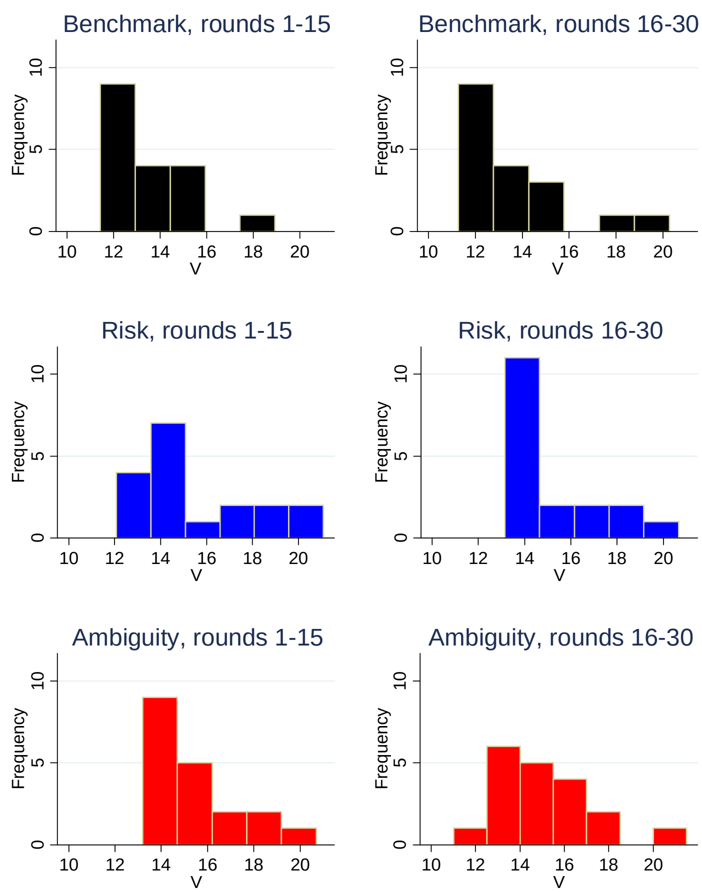

Figure 3.8: Product-Limit mean estimate of the investment trigger for the treatments *Benchmark*, *Risk* and *Ambiguity*, splitting the sample in two subperiods, rounds 1-15 and 16-30.

Benchmark

Subject	**Rounds 1-15**	**Rounds 16-30**
	V^* 95% Conf. Interval	
1	12.89 – 14.51	12.65 – 12.98
2	14.52 – 15.55	14.21 – 14.95
3	12.27 – 13.05	11.87 – 13.00
4	13.11 – 15.14	13.32 – 14.63
5	17.04 – 18.30	17.59 – 19.01
6	11.46 – 12.35	11.67 – 12.54
7	12.20 – 12.76	12.21 – 12.86
8	13.44 – 14.53	13.35 – 15.14
9	11.24 – 11.62	11.04 – 11.52
10	11.34 – 13.34	11.95 – 16.72
11	12.85 – 16.11	11.61 – 12.31
12	14.91 – 15.46	14.74 – 15.78
13	13.12 – 15.95	15.76 – 21.85
14	11.24 – 12.37	11.46 – 12.45
15	11.77 – 11.97	12.34 – 12.60
16	12.01 – 12.81	12.12 – 12.96
17	11.56 – 12.60	11.68 – 13.69
18	12.60 – 13.65	12.39 – 14.60

Table 3.4: 95 percent confidence interval of the Product Limit mean estimate of the investment trigger for the treatment *Benchmark* in rounds 1-15 and 16-30.

Risk

Subject	Rounds 1-15	Rounds 16-30
	V* 95% Conf. Interval	
1	16.22 – 19.35	16.00 – 18.89
2	12.41 – 15.63	12.19 – 13.53
3	13.98 – 15.20	13.66 – 12.30
4	14.75 – 19.51	16.90 – 21.41
5	13.05 – 14.95	11.38 – 12.78
6	12.62 – 13.78	13.71 – 16.45
7	17.93 – 21.23	18.03 – 22.21
8	12.64 – 18.34	13.28 – 15.51
9	12.92 – 13.98	12.78 – 16.07
10	14.81 – 19.33	16.61 – 21.92
11	13.38 – 14.90	13.57 – 14.21
12	12.43 – 13.91	12.21 – 13.36
13	13.06 – 14.70	13.62 – 15.08
14	14.05 – 14.91	15.48 – 17.64
15	12.40 – 13.90	13.22 – 14.81
16	12.13 – 14.29	12.36 – 13.69
17	16.43 – 21.37	15.53 – 18.32
18	13.68 – 16.82	19.03 – 21.74

Table 3.5: 95 percent confidence interval of the Product Limit mean estimate of the investment trigger for the treatment *Risk* in rounds 1-15 and 16-30.

Ambiguity

Subject	**Rounds 1-15**	**Rounds 16-30**
	V^* 95% Conf. Interval	
1	15.83 – 19.67	10.70 – 11.31
2	14.19 – 14.86	15.51 – 18.79
3	15.09 – 17.54	15.49 – 16.95
4	15.48 – 15.68	14.21 – 15.85
5	14.79 – 17.87	17.21 – 17.96
6	12.60 – 14.89	12.55 – 14.25
7	13.79 – 15.67	14.56 – 17.22
8	13.18 – 14.47	15.57 – 16.66
9	17.80 – 21.00	19.57 – 21.38
10	16.51 – 19.69	14.49 – 17.25
11	14.47 – 14.99	12.85 – 13.45
12	11.96 – 15.81	13.73 – 16.87
13	13.70 – 14.23	13.84 – 14.14
14	13.25 – 15.06	13.64 – 15.90
15	12.72 – 14.33	12.48 – 13.50
16	14.64 – 16.49	13.84 – 14.76
17	12.29 – 14.21	12.04 – 13.07
18	14.56 – 17.72	14.51 – 16.00
19	12.73 – 13.66	12.43 – 13.40

Table 3.6: 95 percent confidence interval of the Product Limit mean estimate of the investment trigger for the treatment *Ambiguity* in rounds 1-15 and 16-30.

across rounds. This intuition is statistically confirmed by a log-rank test. The null hypothesis of equality between CDFs across subsamples is not rejected in all treatments (p-values are 0.4843 for *Benchmark*, 0.3970 for *Risk* and 0.3423 for *Ambiguity*).[14] Then, we construct for each individual a Product-Limit estimate for the average of the investment trigger in the two subperiods. Figure 3.8 plots histograms of the by-subject means for each of the three treatments. The figure reveals a surprisingly stable investment pattern in *Benchmark*. Treatments *Risk* and *Ambiguity* display a bit more variability between subperiods but without a clear trend towards an anticipation or a delay of investment. In fact, Table 3.3 shows that means computed on by-subject data do not substantially differ across subperiods. In all three treatments, a pair wise Mann-Whitney test does not reject the null hypothesis of equality of means between subperiods. Also, the main results of the previous section are confirmed. The null hypothesis of equality of means in *Benchmark* and *Risk* and *Benchmark* and *Ambiguity* is rejected at 1% level of significance in both subperiods.

Finally, Tables 3.4, 3.5, and 3.6 show, for each subject, the 95 percent confidence interval of the Product Limit estimate of the mean of the investment trigger in the two subperiods. In the large majority of the cases, the estimated trigger is concentrated around the mean and the investment strategy does not substantially change over time.

3.4.2 The effect of ambiguity: robustness

According to Proposition 3.5, our data imply that subjects are ambiguity seeking. This contrasts with the result of ambiguity aversion commonly found in static Ellsberg-style experiments. In this section we test whether ambiguity seeking is a robust feature of our experimental setting. To do so, we run an additional treatment, named *Mild Ambiguity*, in which we depart from the common practice to provide no information about the probability distribution in the ambiguous scenario. In this new treatment, we set the parameter values for the growth and expiration rates as in

[14] A number of randomly chosen subjects were asked at the end of the experiment about their investment strategy. The majority of them, replied starting with "I was aiming at...", implying that they were actually playing a trigger strategy, as predicted by the model.

	High exp. rate	Low exp. rate	$1-\overline{p}$
Benchmark	10%	5%	50%
Mild Ambiguity	10%	5%	20%-80%

Table 3.7: Parameterization for the treatments *Benchmark* and *Mild Ambiguity*.

Benchmark and *Ambiguity* but we tell experimental subjects that the probability of the expiration rate being high lies somewhere in between 20% and 80%.[15] According to Proposition 3.4, If subjects are ambiguity seeking we expect the following experimental hypothesis to be confirmed by the data.

Hypothesis 3 *The investment trigger in Mild Ambiguity is larger than the one in Benchmark.*

The relevant information about the two treatments of interest for this section is summarized in Table 3.7.

A group of 21 subjects participated in *Mild Ambiguity*. Subjects played 20 practice rounds as in *Benchmark* and the investment game was repeated for 30 rounds. This results in a total of 630 observations. There were no cases in which the option to invest expired when $V_t < C$. Therefore, no observation was dropped. In 256 cases (40% of the total) the option expired before subjects could invest. The analysis follows the same steps as above. First, we consider observations to be i.i.d. and estimate the CDFs, including the censored observations, by using the Product-Limit estimator. The estimated CDF curves are shown in Figure 3.9. The figure shows that the CDF in *Mild Ambiguity* lies slightly to the right of the one in *Benchmark*. A pairwise log-rank test rejects the null hypothesis of equality between the two treatments at 10% level of significance ($p = 0.0611$). We account for within subject dependence by constructing a Product-Limit estimate for the average investment trigger of each subject. Figure 3.10 shows histograms of the by-subject means for each of the three treatments, while Table 3.8 reports means and standard errors of the pooled and by-subject data. The mean in *Mild Ambiguity* is larger than

[15] In practice, the true probability was 50% as in *Benchmark* and *Ambiguity*.

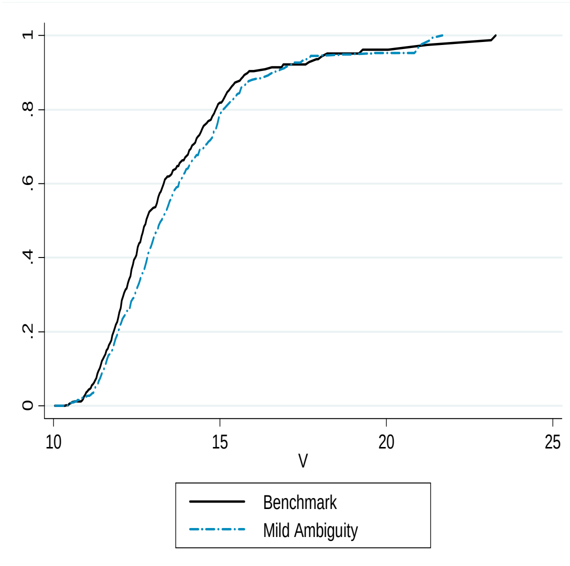

Figure 3.9: Product-Limit estimate of CDFs for the investment trigger for the treatments *Benchmark* (solid line) and *Mild Ambiguity* (dash dotted).

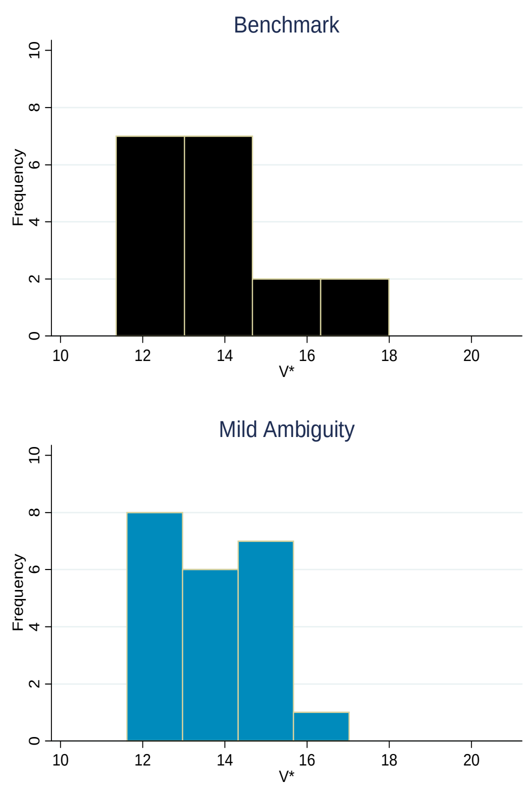

Figure 3.10: Product-Limit mean estimate of the investment trigger for the treatments *Benchmark* and *Mild Ambiguity*.

	Benchmark	**Mild Ambiguity**
	Mean ± Std.Err	
Pooled PL	13.59 ± 0.15	13.83 ± 0.12
By-subject PL	13.61 ± 0.42	13.83 ± 0.29

Table 3.8: Product-Limit estimate of the mean of the investment trigger. The Pooled PL row shows the estimates assuming i.i.d. observations. The by-subjects PL row shows the Product-Limit mean estimates across individual subjects.

the mean in *Benchmark* but the difference is less clear-cut. A pairwise Mann–Whitney test cannot reject the null hypothesis of equality between means ($p = 0.3553$). Overall, we find a weak confirmation of the result of ambiguity seeking.

3.5 Discussion and concluding remarks

In this work we study the effects of risk and ambiguity on the timing of option exercise. We provide a theoretical and an empirical contribution. On the theoretical side, we develop a new real options investment model which allows studying the distinct roles of risk and ambiguity on the timing of option exercise. The model predicts that a higher risk delays investment in accord with the standard results from the real options theory. The effect of ambiguity depends on the attitude of the decision maker. If the decision maker is ambiguity averse, ambiguity accelerates investment, while a delay of investment is consistent with ambiguity seeking.

We test the model in a laboratory experiment. Experimental data show that, when risk is higher, investment is delayed. A higher risk means a greater upside potential for the option and implies that investment should occur for a larger value of the underlying. In our model this result relies on the ability of the subjects to learn from repeated observations. The clear confirmation of the theory suggests that the updating rule proposed in the model is a powerful learning mechanism.

Somewhat surprisingly, we find that also ambiguity delays investment. As the theoretical model suggests, this signals that subjects are ambiguity seeking.[16] The delay in investment is remarkably strong when subjects do not have any background information about the relative probability of the two states of the world, as in the standard Ellsberg setting. However, if subjects face a milder degree of ambiguity and have some initial, though vague, information the effect is substantially weakened. When comparing the treatments *Benchmark* and *Mild Ambiguity*, pooled data still confirm an ambiguity seeking attitude but no statistically significant difference is found accounting for within subjects dependence (Table 3.7). Remarkably, we never find any evidence in favor of ambiguity-aversion.

To conclude, it is important to clarify what our experimental evidence says and what it does not say about the effects of ambiguity. As discussed in Section 3.2.4, acceleration (delay) of investment in response to a higher degree of ambiguity is a sign of ambiguity aversion (seeking). If subjects have an adverse attitude towards ambiguity, they are less willing to withstand the uncertainty associated with the continuation region and invest earlier, and the other way around. Thus, our data are interpreted as evidence in favor of an ambiguity seeking attitude. What data do not reveal is the underlying reason for this favorable attitude towards ambiguity. The purpose of our α-MEU specification is limited to identify the qualitative effect of ambiguity on the timing of investment. In this perspective, the ambiguity coefficient α should not be strictly interpreted as the relative probability that subjects attach to the two states of the world. Rather it should be viewed as a reduced form to capture all the possible channels by which ambiguity potentially influences the decision process. For example, ambiguity seeking could emerge as an effect, not specified in the model, of the interaction between ambiguity and learning.[17] Does

[16] Findings consistent with a non-adverse attitude towards ambiguity have already been found in previous experiments. For example, the evidence reported in Heath and Tversky (1991) reveals that subjects prefer ambiguous bets to risky ones when evaluating situations in which they feel competent or knowledgeable (other examples of non adverse attitude towards ambiguity are found in Cohen and Hansel (1959) Howell (1971), and Ivanov (2010)). Furthermore, ambiguity seeking is found in the majority of experiments conducted in the loss domain (for a list of references, see Wakker (2010), Chapter 12.7).

[17] We assumed that both in the risk and in the ambiguous case subjects update their belief according to Bayes' rule. This means that learning proceeds at the same speed in both scenarios. For some reasons unspecified in the model, it could be that

ambiguity induces optimism as the "literal" interpretation of the model suggests? Does ambiguity seeking depend on the effect of ambiguity on the learning process? Does it depend on another, yet uncovered, behavioral mechanism? The answer to these interesting questions would require a different experimental framework and we leave it as a challenge for future research.

3.A Appendix

3.A.1 Proof of Proposition 3.1

The value of the option to invest satisfies:

$$\begin{aligned}
F_t &= \max_{t^*\geq t}\left[(V_t - C)\, e^{-(r+\lambda)(t^*-t)}\right] && (3.17)\\
&= \max\left\{V_t - C,\ \max_{t^*\geq t+dt}(V_t - C)\, e^{-(r+\lambda)(t^*-t)}\right\} && (3.18)\\
&= \max\left\{V_t - C,\ e^{-(r+\lambda)dt}\max_{t^*\geq t+dt}(V_t - C)\, e^{-(r+\lambda)(t^*-t-dt)}\right\} && (3.19)\\
&= \max\left\{V_t - C,\ e^{-(r+\lambda)dt}\ F_{t+dt}\right\} && (3.20)\\
&= \max\left\{V_t - C,\ e^{-(r+\lambda)dt}\ (F_t + dF_t)\right\} && (3.21)\\
&= \max\left\{V_t - C,\ [1-(r+\lambda)\,dt]\,(F_t + dF_t)\right\}. && (3.22)\\
&= \max\left\{V_t - C,\ F_t +\ dF_t - (r+\lambda)\,F_t dt\right\}. && (3.23)
\end{aligned}$$

The equalities derive from: definition of F_t, (3.17); dividing the problem between investing now, time t, or postponing the investment decision at time $t + dt$, (3.18); definition of F_{t+dt}, (3.20); approximating e^{-rdt} by $1 - (r + \lambda)\,dt$, (3.22); eliminating terms of order higher than dt, (3.23).

In the continuation region, it holds that

$$(r + \lambda)\, F_t dt = dF_t. \tag{3.24}$$

The value of the investment opportunity depends of the current project value V_t, and we rewrite it as $F_t = F\,(V_t)$. Using (3.1), equation (3.24)

learning proceeds faster under ambiguity. If this is the case, subjects would become more rapidly confident that the true expiration rate is low and would delay the exercise the investment option.

can be rewritten as

$$(r+\lambda) F(V_t) = \mu V_t F'(V_t). \tag{3.25}$$

The general solution of (3.26) is $F(V_t) = AV_t^{\beta}$, where $\beta = (\lambda + r)/\mu$. The coefficient A and the investment trigger V_K^* are defined by the boundary conditions

$$F(V_K^*) = V_K^* - C \tag{3.26}$$

$$F'(V_K^*) = 1 \tag{3.27}$$

Condition (3.26) is the value-matching condition and implies that at the trigger V_K^* the value of the option to invest equals the value of the project minus the sunk investment cost. Condition (3.27) is the so-called smooth-pasting condition and implies that the trigger V_K^* is optimally chosen by the DM. Solving the system (3.26)-(3.27) yields the solution in (3.3).

3.A.2 Proof of Proposition 3.2

Here we derive the expression for the investment trigger (3.10). The value of the option to invest satisfies:

$$\begin{aligned}
F_t &= p_t \max_{t^*\geq t}\left[(V_t - C)\, e^{-(r+\lambda_L)(t^*-t)}\right] + (1-p_t)\max_{t^*\geq t}\left[(V_t - C)\, e^{-(r+\lambda_H)(t^*-t)}\right] \\
&= \max\left\{V_t - C,\ p_t \max_{t^*\geq t+dt}\left[(V_t - C)\, e^{-(r+\lambda_L)(t^*-t)}\right] + \right. \\
&\quad \left. (1-p_t)\max_{t^*\geq t+dt}\left[(V_t - C)\, e^{-(r+\lambda_H)(t^*-t)}\right]\right\} \\
&= \max\left\{V_t - C,\ p_t e^{-(r+\lambda_L)dt}\max_{t^*\geq t+dt}\left[(V_t - C)\, e^{-(r+\lambda_L)(t^*-t-dt)}\right] + \right. \\
&\quad \left. + (1-p_t)\, e^{-(r+\lambda_H)dt}\max_{t^*\geq t+dt}\left[(V_t - C)\, e^{-(r+\lambda_H)(t^*-t-dt)}\right]\right\} \\
&= \max\left[V_t - C,\ p_t e^{-(r+\lambda_L)dt}\ F_{t+dt} + (1-p_t)\, e^{-(r+\lambda_H)dt}\ F_{t+dt}\right] \\
&= \max\left[V_t - C,\ p_t\left[1-(r+\lambda_L)\, dt\right]\ F_{t+dt} + (1-p_t)\left[1-(r+\lambda_H)\, dt\right] F_{t+dt}\right] \\
&= \max\left\{V_t - C,\ p_t\left[1-(r+\lambda_H)\, dt\right]\ (F_t + dF_t)\right. \\
&\quad \left. + (1-p_t)\left[1-(r+\lambda_H)\, dt\right](F_t + dF_t)\right\} \\
&= \max\left\{V_t - C,\ F_t +\ dF_t - \left[r + (1-p_t)\,\lambda_H + p_t\lambda_L\right] F_t dt\right\}.
\end{aligned} \tag{3.28}$$

Equation (3.28) implies that in the continuation region F_t satisfies

$$[r + (1 - p_t)\lambda_H + p_t\lambda_L] F_t dt = dF_t. \tag{3.29}$$

The value of the investment opportunity F_t depends on both the value of the project V_t and the current belief p_t. Thus, we rewrite it as $F_t = F(V_t, p_t)$. However, since both V_t and p_t are deterministic functions of time, the dimensionality of the problem can be reduced to only one state variable.

Define $\pi_t = \pi(p_t) = p_t/(1 - p_t)$. The instantaneous change in π_t is given by $d\pi_t = \pi'(p_t)\,dp = (\lambda_H - \lambda_L)\,\pi_t dt$, which implies that

$$\pi_t = \pi_0 e^{(\lambda_H - \lambda_L)t}, \tag{3.30}$$

where $\pi_0 = \overline{p}/(1-\overline{p})$. Solving (3.30) for time t yields $t = (\lambda_H - \lambda_L)^{-1}\ln(\pi_t/\pi_0)$. Using (3.2), V_t can be written as

$$V_t = V_0\left(\frac{\pi_t}{\pi_0}\right)^{\frac{\mu}{\lambda_H - \lambda_L}}. \tag{3.31}$$

Solving (3.31) for π_t yields

$$\pi_t = \Omega_R V_t^{\frac{\lambda_H - \lambda_L}{\mu}}, \tag{3.32}$$

where $\Omega_R = \pi_0 V_0^{\frac{\lambda_L - \lambda_H}{\mu}}$.

Finally, using the fact that $p_t = \pi_t/(1+\pi_t)$, DM's belief can be rewritten

$$p_t = p(V_t) = \frac{\Omega_R V_t^{\frac{\lambda_H - \lambda_L}{\mu}}}{1 + \Omega_R V_t^{\frac{\lambda_H - \lambda_L}{\mu}}}. \tag{3.33}$$

Equation (3.33), allows us to write the option value as a function of V_t only, $F(V_t, p_t) = F(V_t)$. Thus, equation (3.29) becomes

$$[r + p(V_t)\lambda_L + (1 - p(V_t))\lambda_H] F(V_t) = \mu V F'(V_t). \tag{3.34}$$

Solution of (3.34) implies

$$F(V_t) = \frac{B V_t^{\frac{r + \lambda_H + \lambda_L}{\mu}}}{V_t^{\frac{\lambda_L}{\mu}} + \Omega_R V_t^{\frac{\lambda_H}{\mu}}}, \tag{3.35}$$

where B is a constant.

The investment trigger V_R^* and the constant B are defined by the boundary conditions:

$$F\left(V_R^*\right) = V_R^* - C \tag{3.36}$$

$$F'\left(V_R^*\right) = 1 \tag{3.37}$$

Conditions (3.36)-(3.37) have the same interpretation of (3.26)-(3.27). Plugging (3.36) in (3.37) and rearranging yields

$$V_R^* = \frac{\beta\left(V_R^*\right)}{\beta\left(V_R^*\right) - 1} C. \tag{3.38}$$

The expression for $\beta_R\left(V_t\right)$ is given in Proposition 3.2.

Condition (3.4) ensures that there exists a unique trigger that separates the stopping and continuation regions (see Appendix B, Chapter 3, in Dixit and Pindyck (1994)). ■

3.A.3 Proof of Proposition 3.3

Here we verify the effects of an increase in risk on the optimal investment trigger. An increase in risk is represented by a mean preserving spread between λ_H and λ_L at the initial date, that is a rise in $\Delta\lambda = \lambda_H - \lambda_L$ which leaves $\overline{p}\lambda_L + (1-\overline{p})\lambda_H$ unaffected. Define the function $f_t = \beta_R\left(V_t\right) / \left(\beta_R\left(V_t\right) - 1\right) C$.

Optimality implies that the investment option is exercised whenever $V_t - f_t \geq 0$. In the waiting region it holds $V_t - f_t < 0$, while at the investment trigger V_R^* it holds $V_R^* - f_t = 0$. Using the implicit function theorem, the effect of an increase in risk on the investment trigger can be found as $dV_R^*/d\Delta\lambda = \frac{\partial f_t/\partial\Delta\lambda}{1-\partial f_t/\partial V_t}$, where the derivatives with respect to $\Delta\lambda$ are for $\overline{p}\lambda_L + (1-\overline{p})\lambda_H$ held constant.

The proof proceeds in two steps. First, we claim that at $V_t = V_R^*$ it must hold $1 - \partial f_t/\partial V_t \geq 0$. Then, we prove that $\partial f_t/\partial\Delta\lambda > 0$, so that $dV_R^*/d\Delta\lambda > 0$.

Suppose that the initial value of the project is below the investment trigger, $V_0 < V_R^*$, and the decision maker is in the waiting region, where it holds $V_t < f_t$. Since $\beta_R\left(V_t\right)$ is decreasing in V_t, as V_t increases f_t also increases and the decision maker invests as soon as $V_R^* = f_t$. Thus, at the unique trigger point V_R^*, f_t crosses (or meets tangentially) the 45 degrees line V_t from above. Thus, the slope of f_t must be lower than or equal to one, i.e., $\partial f_t / \partial V_t \leq 1$.

Consider, now, the sign of $\partial f_t / \partial \Delta\lambda$. It is useful to rewrite the expression for $\beta_R(V_t)$ as

$$\beta_R(V_t) = \beta_R(V_0) - \frac{\Delta\lambda}{\mu}\left(p(V_t) - \overline{p}\right), \tag{3.39}$$

where $\beta_R(V_0) = \left[r + \overline{p}\lambda_L + (1-\overline{p})\lambda_H\right]/\mu$. Differentiating with respect to $\Delta\lambda$ yields

$$\frac{\partial\beta_R(V_t)}{\partial\Delta\lambda} = \frac{\partial\beta_R(V_0)}{\partial\Delta\lambda} - \frac{p(V_t)}{\mu} - \frac{\Delta\lambda}{\mu}\frac{\partial p(V_t)}{\partial\Delta\lambda}. \tag{3.40}$$

Since we are considering a mean preserving spread between λ_H and λ_L at date 0, it holds by definition that $\partial\beta_R(V_0)/\partial\Delta\lambda = 0$. Combined with the fact that $\partial p(V_t)/\partial\Delta\lambda > 0$ and $p(V_t) > 0$, this implies that the sign of $\partial\beta_R(V_t)/\partial\Delta\lambda$ is unambiguously negative. Since $\partial f_t/\partial\beta_R(V_t) < 0$, it holds that it hold that $\partial f_t/\partial\Delta\lambda = \frac{\partial f_t/\partial\beta_R(V_t)}{\partial\beta_R(V_t)/\partial\Delta\lambda} > 0$. The claim in the proposition follows. ■

3.A.4 Proof of Proposition 3.4

Under the α-MEU specification, the DM evaluates the investment opportunity according to a convex combination α and $1-\alpha$ of the worst case and best case beliefs p_t^- and p_t^+, respectively. The value of the option to invest satisfies:

$$
\begin{aligned}
F_t &= \alpha \left\{ p_t^- \max_{t^* \geq t} \left[(V_t - C)\, e^{-(r+\lambda_L)(t^*-t)} \right] \right. \\
&\quad \left. + \left(1 - p_t^-\right) \max_{t^* \geq t} \left[(V_t - C)\, e^{-(r+\lambda_H)(t^*-t)} \right] \right\} + \\
&\quad (1-\alpha) \left\{ p_t^+ \max_{t^* \geq t} \left[(V_t - C)\, e^{-(r+\lambda_L)(t^*-t)} \right] + \right. \\
&\quad \left. \left(1 - p_t^+\right) \max_{t^* \geq t} \left[(V_t - C)\, e^{-(r+\lambda_H)(t^*-t)} \right] \right\} \\
&= \max \left\{ V_t - C, \alpha\ p_t^- \max_{t^* \geq t+dt} \left[(V_t - C)\, e^{-(r+\lambda_L)(t^*-t)} \right] + \right. \\
&\quad \alpha \left(1 - p_t^-\right) \max_{t^* \geq t+dt} \left[(V_t - C)\, e^{-(r+\lambda_H)(t^*-t)} \right] + \\
&\quad (1-\alpha)\ p_t^+ \max_{t^* \geq t+dt} \left[(V_t - C)\, e^{-(r+\lambda_L)(t^*-t)} \right] + \\
&\quad \left. (1-\alpha) \left(1 - p_t^+\right) \max_{t^* \geq t+dt} \left[(V_t - C)\, e^{-(r+\lambda_H)(t^*-t)} \right] \right\} \\
&= \max \left\{ V_t - C,\ \alpha p_t^- e^{-(r+\lambda_L)dt}\ F_{t+dt} + \right. \\
&\quad \alpha \left(1 - p_t^-\right) e^{-(r+\lambda_H)dt}\ F_{t+dt} + (1-\alpha)\ p_t^+ e^{-(r+\lambda_L)dt}\ F_{t+dt} + \\
&\quad \left. (1-\alpha) \left(1 - p_t^+\right) e^{-(r+\lambda_H)dt}\ F_{t+dt} \right\} \\
&= \max \left\{ V_t - C,\ \alpha p_t^- \left[1 - (r + \lambda_L)\, dt\right]\ F_{t+dt} + \right. \\
&\quad \alpha \left(1 - p_t^-\right) \left[1 - (r + \lambda_H)\, dt\right]\ F_{t+dt} + \\
&\quad (1-\alpha)\, p_t^+ \left[1 - (r + \lambda_L)\, dt\right]\ F_{t+dt} + \\
&\quad \left. (1-\alpha) \left(1 - p_t^+\right) \left[1 - (r + \lambda_H)\, dt\right] F_{t+dt} \right\} \\
&= \max \left\{ V_t - C,\ \alpha \left(dF_t + F_t\right) - \right. \\
&\quad \alpha \left[r + \left(1 - p_t^-\right) \lambda_H + p^- \lambda_L \right] F_t dt + (1-\alpha) \left(dF_t + F_t\right) - \\
&\quad \left. (1-\alpha) \left[r + \left(1 - p_t^+\right) \lambda_H + p_t^+ \lambda_L \right] F_t dt \right\}. \qquad (3.41)
\end{aligned}
$$

Rearranging gives the expression in (3.13).

In the waiting region it holds

$$\left[\alpha R_t^- + (1-\alpha)\, R_t^+\right] F_t dt = dF_t, \qquad (3.42)$$

where the expressions for R_t^- and R_t^+ are given in equation (3.14).

The option value F_t depends on the three state variables V_t, p_t^- and p_t^+, and can be written as $F_t = F\left(V_t, p_t^-, p_t^+\right)$. As in the Appendix 3.A.2, we can express the probabilities p_t^- and p_t^+ as a function of V_t:

$$p^-(V_t) = \frac{\Omega^- V_t^{\frac{\lambda_H - \lambda_L}{\mu}}}{1 + \Omega^- V_t^{\frac{\lambda_H - \lambda_L}{\mu}}} \text{ and } p^+(V_t) = \frac{\Omega^+ V_t^{\frac{\lambda_H - \lambda_L}{\mu}}}{1 + \Omega^+ V_t^{\frac{\lambda_H - \lambda_L}{\mu}}}, \tag{3.43}$$

where $\Omega^- = \frac{\overline{p}-\varepsilon}{1-(\overline{p}-\varepsilon)} V_0^{\frac{\lambda_L - \lambda_H}{\mu}}$ and $\Omega^+ = \frac{\overline{p}+\varepsilon}{1-(\overline{p}+\varepsilon)} V_0^{\frac{\lambda_L - \lambda_H}{\mu}}$. Then, we can write $F\left(V_t, p_t^-, p_t^+\right) = F(V_t)$. Equation (3.42) implies that, in the waiting region, the value of the option satisfies the following ordinary differential equation:

$$[r + p_A(V_t)\lambda_L + (1 - p_A(V_t))\lambda_H] F(V_t) = \mu V_t F'(V_t), \tag{3.44}$$

where $p_A(V_t) = \alpha p^-(V_t) + (1-\alpha) p^+(V_t)$. The general solution of (3.44) is

$$F(V_t) = \frac{D V_t^{\frac{r + \lambda_H + \lambda_L}{\mu}}}{\left(V_t^{\frac{\lambda_L}{\mu}} + \Omega^- V_t^{\frac{\lambda_H}{\mu}}\right)^{\alpha} \left(V_t^{\frac{\lambda_L}{\mu}} + \Omega^+ V_t^{\frac{\lambda_H}{\mu}}\right)^{1-\alpha}}, \tag{3.45}$$

where D is a positive constant.

The investment trigger V_A^* and the constant D are defined by the boundary conditions.

$$F(V_A^*) = V_A^* - C \tag{3.46}$$

$$F'(V_A^*) = 1 \tag{3.47}$$

Solving the system (3.46)-(3.47) yields the expression for the investment trigger:

$$V_A^* = \frac{\beta_A(V_A^*)}{\beta_A(V_A^*) - 1} C \tag{3.48}$$

where the expression for $\beta_A(V_t)$ is given in Proposition 3.4.

Condition (3.4) ensures that there exists a unique trigger that separates the stopping and continuation regions. ■

3.A.5 Proof of Proposition 3.5

Define the function $g_t = \beta_A(V_t) / (\beta_A(V_t) - 1) C$. At $V_t = V_A^*$ it holds $V_A^* - g_t = 0$. Using the implicit function theorem, the marginal effect of a change in the ambiguity attitude on the investment trigger is given by

$dV_A^*/d\alpha = \frac{\partial g_t/\partial\alpha}{1-\partial g_t/\partial V_t}$. Following the same argument of Appendix 3.A.3, it can be proved that $1-\partial g_t/\partial V_t \geq 0$. Furthermore, from $d\beta_A(V_t)/dp_A(V_t)$ and $dp_A(V_t)/d\alpha$, it follows that $\partial g_t/\partial\alpha < 0$. Then, $dV_A^*/d\alpha < 0$ and the claim in the proposition follows. ■

3.A.6 Risk aversion

Following the standard real options approach, we analyzed the investment problem under the assumption of risk neutrality. Here we generalize our arguments allowing for the DM being risk averse. We first solve the stopping problem with known expiration rate of Section 3.2.1. Then, the reasoning is easily extended to find the solution for the risky and ambiguous cases with unknown expiration rate of Sections 3.2.3 and 3.2.4. We show that the fundamental predictions of the model about the effects of risk and ambiguity on the investment timing are not affected by risk aversion.

Let us consider the setting with a known expiration rate described in Section 3.2.1 and assume that DM's preferences are described by a constant relative risk aversion utility function (CRRA): $U = w^{1-\gamma}/1-\gamma$, where γ ($\gamma > 0$ and $\gamma \neq 1$[18]) is the coefficient of relative risk aversion and w is DM's "wealth". Wealth is either the final payoff $V_t - I$, after the investment is undertaken, or the option to invest at an optimally chosen time in the future if the investment is yet not undertaken. We conjecture that in the continuation region the utility function has the following form:

$$U(V_t) = \frac{G(V_t)^{1-\gamma}}{1-\gamma}. \tag{3.49}$$

The function $G(V_t)$ can be interpreted as the certainty equivalent wealth deriving from the investment opportunity (see Miao and Wang (2007)). Define by $\widetilde{V}_K^*$ the optimal investment trigger. The above discussion implies that

$$U(V_t) = \begin{cases} \frac{(V_t-I)^{1-\gamma}}{1-\gamma} & \text{if } V_t \geq \widetilde{V}_K^* \\ \frac{G(V_t)^{1-\gamma}}{1-\gamma} & \text{if } V_t < \widetilde{V}_K^* \end{cases} \tag{3.50}$$

In the continuation region, for $V < V^*$, DM's utility satisfies:

$$(r+\lambda)U(V_t) = \mu V U'(V_t). \tag{3.51}$$

[18] For $\gamma = 1$ the CRRA utility is $U = \ln(w)$.

subject to the boundary conditions:

$$U\left(\widetilde{V}_K^*\right) = \frac{(\widetilde{V}_K^* - 1)^{1-\gamma}}{1-\gamma}, \tag{3.52}$$

$$U'\left(\widetilde{V}_K^*\right) = (\widetilde{V}_K^* - 1)^{-\gamma}. \tag{3.53}$$

Conditions (3.51) and (3.52) have the same interpretation as (3.26)-(3.27). Substituting (3.49) in (3.51) and simplifying, we obtain

$$\frac{(r+\lambda)}{1-\gamma} G(V_t) = \mu V_t G'(V_t). \tag{3.54}$$

Conditions (3.51) and (3.52) can be simplified to $G(V_K^*) = V_K^* - I$ and $G'(V_K^*) = 1$, respectively. Solution of (3.54) implies $G(V_t) = A_0 V^{\frac{(r+\lambda)}{\mu(1-\gamma)}}$, where A_0 is a constant. Using the appropriate boundary conditions, the investment trigger can be found as:

$$\widetilde{V}_K^* = \frac{\eta}{\eta - 1} I, \tag{3.55}$$

where

$$\eta = \frac{r+\lambda}{\mu(1-\gamma)}. \tag{3.56}$$

From (3.55) and (3.56), it is immediate to see that risk aversion leads to an earlier exercise of the investment option, i.e. $d\widetilde{V}_K^*/d\gamma < 0$. The explanation is that the option value $G(V_t)$ is decreasing in γ while the final payoff is independent of risk aversion. This implies that a more risk averse DM is less willing to withstand the uncertainty associated with the continuation region and will exercise the investment option sooner.[19]

Consider, now, the risky scenario described in Section 3.2.3. DM's utility is the same as in (3.52) where the $\widetilde{V}_K^*$ must be substituted the appropriate (and yet to be determined) investment trigger, indicated by $\widetilde{V}_R^*$. Following the same steps as above, it is easy to show that the option value $G(V_t)$ satisfies

$$\frac{r + p(V_t)\lambda_L + (1 - p(V_t))\lambda_H}{1-\gamma} G(V_t) = \mu V G'(V_t), \tag{3.57}$$

[19] This is the same result found in Miao and Wang (2007) when the final payoff is lump sum. They also show that, when the final payoff is given by a flow of an uncertain income, the result is reversed.

under the boundary conditions to $G(V_K^*) = V_K^* - I$ and $G'(V_K^*) = 1$. The expression for $p(V_t)$ is given in (3.33). The solution for the investment trigger is:

$$\widetilde{V}_R^* = \frac{\eta_R\left(\widetilde{V}_R^*\right)}{\eta_R\left(\widetilde{V}_R^*\right) - 1} C, \tag{3.58}$$

where $\eta_R(V_t) = [r + p(V_t)\lambda_L + (1 - p(V_t))\lambda_H] / \mu(1-\gamma)$.

Analogous steps leads to the following expression for the investment trigger of the ambiguous scenario desribed in 3.2.4:

$$\widetilde{V}_A^* = \frac{\eta_A\left(\widetilde{V}_A^*\right)}{\eta_A\left(\widetilde{V}_A^*\right) - 1} C, \tag{3.59}$$

where $\eta_A(V) = [r + p_A(V_t)\lambda_L + (1 - p_A(V_t))\lambda_H] / \mu(1-\gamma)$ and $p_A(V_t) = \alpha p^-(V_t) + (1-\alpha) p^+(V_t)$.

As it is easy to check, expressions (3.58) and (3.59) imply that the degree of risk aversion γ does not affect comparative static conclusions about the effects of risk and ambiguity on the timing of investment.

3.B Instructions

We report the instructions for the treatment *Benchmark*. Instructions for the other two treatments are analogous, and only parameter values are changed.

3.B.1 Instructions for the investment game

You will participate in an experiment on investment decisions where, depending on your performance, you can win a considerable amount of money and cannot make losses.

THE EXPERIMENT

The idea of the experiment is the following. You have the opportunity to invest in a project by paying a cost equal to 10 Euros. By investing you win the value of the project minus the investment cost.

The value of the project grows by 3% per second but the option to invest can expire at a positive rate. If the option to invest expires you get nothing.

Your have to decide when to invest.

You will not be informed about the true expiration rate. However, you know that:

- the expiration rate can be either HIGH (10%) or LOW (5%), and
- the probability that the expiration rate is HIGH is 50%.

The investment game will be repeated for 30 rounds. The true expiration rate is randomly chosen by the computer at the beginning of each round.

SCREEN INFORMATION

At the beginning of each round the initial screen shows the value of the HIGH expiration rate, the value of the LOW expiration rate, and the probability that the expiration rate is HIGH. You find a button "OK" at bottom-right of the initial screen. When you are ready to start, click "OK". If you click "OK" a new screen appears and the investment game starts. In the new screen you can see the investment cost and the value of the project that grows over time. There is a button "INVEST" at bottom-right of the screen. When you want to invest, click "INVEST". If you invest before the option to invest expires, a message appears to tell you how much you win. If the option to invest expires before you invest the message "You did not invest" appears. The initial value of the project is lower than the cost of investment. The computer program forbids you to make losses. If you try to invest when the value of the project is lower than the cost, the message "You cannot invest if $Value$ is lower than $Cost$" appears. If you have further questions, raise your hand to call one of the experimenters.

PAYMENT

At the end of the experiment, you will be paid 5 Euros as a participation fee plus the amount of money that you win in *one* of the rounds. For example, assume that the experiment lasts 4 rounds and you win 2, 0, 7 and 20 Euros in each of these rounds. If the "payment round" is the third, you will get 5 Euros as participation fee plus 7 Euros. The payment round is chosen at random at the end of the experiment.

GOOD LUCK!

Chapter 4

Learning Investment

4.1 Introduction

Technologies in many industries are characterized by learning curves. While producing, firms exploit a process of learning-by-doing that leads to increased efficiency and lower production costs in the future. Investment in these technologies often requires substantial up-front sunk costs and may generate losses in earlier stages. These costs are compensated by the benefits of learning and potential future profits. Whether a new technology turns ultimately profitable depends on the development of its market, which is usually subject to substantial uncertainty. Some recent examples of widely publicized learning investments, i.e. those that are intended to move down the learning curve, include hybrid cars and solar photovoltaic cells.[1]

This article aims to study learning investment under uncertainty. Specifically, we investigate the optimal timing and scale of investment when demand is uncertain and marginal costs decrease with cumulative production. Whereas the literature on investment under uncertainty mainly focuses on the optimal timing of investment, we also investigate the choice

[1] There is ample empirical evidence documenting the presence of learning effects in many other industries. See Wright (1936), Hirsh (1952), Webbink (1977), Zimmerman (1982), Lieberman (1984), Argote, Beckman and Epple (1990), Gruber (1992), Bahk and Gort (1993), Thompson (2001), and Thronton and Thompson (2001) among others.

of optimal capacity. This approach is dictated by the fact that scale considerations play a primary role in the presence of the learning curve. A larger capacity allows a higher per-period production rate and a faster reduction of marginal production costs.[2]

When the scale of investment is flexible but the timing is not, we find that the presence of the learning curve implies that firms should invest in a larger capacity. On the other hand, when the timing is flexible but the scale is fixed, the learning curve accelerates the investment. These two observations suggest that investment should occur early and on a large scale to maximize the benefit of learning. However, when timing and scale of investment are simultaneously chosen, a firm faces a trade-off. Investing early, that is, investing at the moment that levels of demand or productivity are still low, implies that only small scale projects are optimal. At the same time, a large scale investment typically requires higher demand or productivity and entails a longer waiting time resulting in foregone profits. Therefore, an optimal investment strategy requires finding a balance between timing and scale that allows firms to benefit from the learning curve, but which is not too costly in the short run.

The resolution of the timing-scale trade-off depends on the steepness of the learning curve. Under slow learning, investment occurs relatively late and on a larger scale, whereas under fast learning it occurs early and on a smaller scale. In the latter case, firms do not need large production rates to substantially reduce marginal costs. Hence, it is optimal to invest soon and install a small capacity. The opposite holds under slow learning, because then optimality implies that a firm should install a larger capacity to reduce marginal costs sufficiently within a given amount of time. Given the larger project size, investment is delayed. It turns out that, where timing is accelerated, scale is inversely U-shaped in the steepness of the learning curve.

To take advantage of learning benefits, firms may undertake learning investments even when current revenue rates are below costs. We show that the optimal investment rule implies that losses at the moment of

[2] Capozza and Yuming (1994), Bar-Ilan and Strange (1999), and Dangl (1999) also determine the optimal scale of investment next to the investment timing decision. In contrast to our paper, they do not consider learning effects.

investment are accepted even for relatively flat learning curves. What matters from an economic point of view is how large accumulated losses are before firms break even. Our analysis indicates that, first, the present value of expected initial losses is large. Second, the amount of initial losses is the largest for moderate learning rates. For steep learning curves, the initial level of losses is similar but, because of rapid learning, the break-even point is reached sooner. Third, the losses incurred in early production stages can easily dwarf the initial investment outlays to set up the production facility. Overall, these findings indicate that learning investments can be financially very demanding for firms. This is especially true for technologies with intermediate learning curves.

Learning investment may be particularly exposed to downside risk. New technologies may be superseded by newer technologies, turn out to be unmarketable or flawed. To analyze how downside risk affects optimal investment, we extend the model by introducing the possibility that the project fails and vanishes at a random time. We show that learning investment is very sensitive to this type of risk. Investment is significantly delayed and scale increases with the occurrence of even small levels of downside risk. In contrast, timing and scale of non-learning investment are very insensitive to this type of risk. Furthermore, the value of investment projects with learning curves is decreased more by downside risk. Interestingly, the effects of risk on learning investment are strong for moderate learning curves and steeper curves do not amplify these effect further. The explanation is related to the initial losses associated with learning investment, which are similar for these cases. The threat of the project expiring before any profits materialize, distorts learning investment and prevents long-term benefits of learning from being fully exploited.

Past theoretical research has recognized the learning curve as a key factor behind firms' production policies and competitive strategies. Some important contributions include Spence (1981), Brueckner and Raymon (1983), Fudenberg and Tirole (1983), Dasgupta and Stiglitz (1988), Stefanou, Majd and Pindyck (1989), Cabral and Riordan (1994), Dutta and Prasad (1996), Cabral and Riordan (1997), Auerswald, Kauffman, Lobo, and Shell (2000), Besanko, Doraszelski, Kryukov, and Satterthwaite (2010). However, little attention has been paid to the effects of the learning curve on corporate investment. The closest to our work is the work by Majd and Pindyck (1989), which studies the optimal production rate under the learning curve and uncertain demand. In their continuous-time model a production facility with fixed capacity is given and no investment decision is analyzed. In contrast, we study flexible investment in a new technology facility to show that the learning curve can significantly affect the choice of investment timing and optimal capacity.

This work is organized as follows. Section 4.2 presents the model of investment in the presence of the learning curve, whereas Section 4.3 analyzes the optimal choice of timing and scale of investment. Section 4.4 studies initial losses associated with learning investment, and Section 4.5 introduces jump downside risk and investigates its effects on investment. Finally, Section 4.6 concludes. Proofs are relegated to the Appendix.

4.2 A model of investment with the learning curve

Time is continuous and labelled by $t \in [0, \infty)$. A firm holds an option to develop a production facility with a technology characterized by a learning curve. The exercise of the investment option involves the decisions of when to invest (timing) and how much capital to install (scale). To focus on learning investment, we assume that at the initial time the firm has no capital invested in the technology. Investment is irreversible and is associated with a lumpy up-front cost. A unit of capital costs i, so investment in K units of capital requires an investment expense of $I(K) = iK$. Once in place, the lifetime of the production facility is assumed to be infinite.

Capital at level K is used to produce output. The production technology is characterized by constant returns to scale and each unit of capital produces one unit of output. The firm produces at its capacity determined by the scale of investment.[3] This implies that per-period output q is equal to the level of capital, i.e. $q = K$.

Each unit of output is produced at non-negative marginal costs. The

[3] For simplicity we do not allow for temporary suspension of the production; its role has been studied in Majd and Pindyck (1989).

learning curve allows the firm to decrease these costs with accumulated experience. At each point in time, marginal costs are constant with respect to the rate of output but, starting from an initial level c, they decline with cumulative output Q. At each time t, Q is given by $\int_0^t q_t dt$. To model the learning curve we follow Majd and Pindyck (1989) and set the instantaneous marginal cost equal to

$$c(Q) = ce^{-\gamma Q}, \tag{4.1}$$

where $\gamma > 0$ is an exogenous parameter that determines the intensity of the learning process.[4] A high (low) γ means that the learning curve is steep (flat).

The firm's output is non-storable and sold at a unit market price denoted by P. The instantaneous profit function is then given by

$$\pi = \left(P - ce^{-\gamma Q}\right) K. \tag{4.2}$$

Profits are discounted at rate ρ.

We assume that the price is determined by the inverse demand function[5]

$$P = X - \varphi q, \tag{4.3}$$

where φ is a strictly positive constant, and X is a demand shift parameter that fluctuates according to a geometric Brownian motion with drift μ and variance σ:

$$\frac{dX_t}{X_t} = \mu dt + \sigma dZ_t. \tag{4.4}$$

The drift and the discount rate are related such that $\rho > \mu$.[6]

[4] Whereas we assume that the marginal costs asymptotically approach zero, Majd and Pindyck (1989) assume that the learning process stops as soon as marginal cost reaches a strictly positive lower bound. Employing Majd and Pindyck's specification would not change our qualitative results.

[5] The results presented below are not driven by the model specification. In particular, all the results are also present in another popular specification in these types of models: a price taking firm with decreasing returns to scale technology (P_t is an exogenous diffusion process and the rate of production with capital K is K^α, $\alpha < 1$). The analysis is available from the authors upon request.

[6] If $\rho - \mu \leq 0$, it would be optimal to indefinitely postpone the investment.

The per-period profit can be written as a function of demand shock X, capital stock K, and cumulative output Q:

$$\pi = \left(X - \varphi K - ce^{-\gamma Q}\right) K. \tag{4.5}$$

Once the capital is in place, the facility yields an expected discounted stream of profits equal to

$$V(X,K,Q) = \mathbb{E}\left[\int_0^\infty \left(X_t - \varphi K - ce^{-\gamma Q_t}\right) Ke^{-\rho t}dt \mid X_0 = X,\ Q_0 = Q\right].$$

Given investment scale K, Q_t is equal to Kt. Using this we obtain that

$$V(X,K,Q) = \frac{XK}{\rho - \mu} - \frac{\varphi K^2}{\rho} - \frac{ce^{-\gamma Q}K}{\rho + \gamma K}. \tag{4.6}$$

Note that the stream of production costs are discounted at the "learning adjusted" rate $\rho + \gamma K$. A larger capacity implies a larger per-period production rate, faster learning and, therefore, a lower discounted stream of costs.

4.3 Timing and scale of learning investment

4.3.1 Benchmarks: Fixed timing and fixed scale

In the model the firm simultaneously chooses timing and scale of investment. However, we initially consider two benchmark cases. In the first, the firm can only choose the optimal project size without the option to delay the investment. In the second, it has the flexibility to choose the investment timing while the size is fixed.

Consider the case in which the firm's strategy is limited to the optimal capacity choice. Given the market conditions X, in this scenario the firm chooses the optimal size of investment $\overline{K}$ by solving

$$\max_K \left[V(X,K,0) - iK\right]. \tag{4.7}$$

The optimal capacity is implicitly determined by the standard optimality condition that equates the marginal value of an additional unit of capital with the marginal cost:

$$\frac{X}{\rho - \mu} - \frac{2\varphi\overline{K}}{\rho} - \frac{c}{\rho + \gamma\overline{K}} + \frac{\gamma c\overline{K}}{\left(\rho + \gamma\overline{K}\right)^2} = i. \tag{4.8}$$

The first question that we want to answer is how the speed of the learning process affects the optimal size of the project when investment cannot be delayed. That is, we want to know how γ affects $\overline{K}$. Condition (4.8) cannot be explicitly solved for $\overline{K}$. However, we can show that the following proposition holds.

Proposition 4.1 *When the firm can choose the scale but not the timing of investment, the scale $\overline{K}$ is increasing in γ.*

Proposition 1 implies that a more intense learning process increases the scale of investment. Intuitively, a larger γ means a faster reduction of the marginal costs, a larger marginal value of capital and, therefore, a larger optimal capacity.

Consider now the case of a firm that has an option to choose the optimal timing of investment for a project of fixed size K. The firm observes the evolution of the market conditions X and invests at time t^*, where $t^* = \inf\left\{t : X \geq \overline{X}\right\}$ and $\overline{X}$ is the demand level that triggers the investment. The investment trigger $\overline{X}$ is optimally chosen by the firm.

Denote by $F(X, K)$ the value of the option to invest. Standard arguments (Dixit and Pindyck (1994)) imply that this option satisfies the ordinary differential equation

$$\frac{1}{2}\sigma^2 X F_{XX}(X, K) + \mu X F_X(X, K) - \rho F(X, K) = 0. \tag{4.9}$$

The general solution of (4.9) is $A(K)X^{\beta_1} + B(K)X^{\beta_2}$, where

$$\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}} > 1,$$

$$\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}} < 0.$$

The two coefficients, $A(K)$ and $B(K)$, are to be determined by appropriate boundary conditions. As X reaches zero (its absorbing state), the investment opportunity is foregone forever and the investment option is valueless, i.e. $F(0, K) = 0$. This implies that $B(K) = 0$. The investment trigger $\overline{X}(K)$ and the coefficient $A(K)$ are obtained from the value matching and smooth pasting conditions

$$F(\overline{X}(K), K) = V(\overline{X}(K), K, 0) - iK, \tag{4.10}$$

$$F_X(\overline{X}(K), K) = V_X(\overline{X}(K), K, 0). \tag{4.11}$$

Substitution of (4.6) into (4.10) and (4.11) eventually yields

$$\overline{X}(K) = \frac{\beta_1 (\rho - \mu)}{\beta_1 - 1} \left(\frac{\varphi K}{\rho} + \frac{c}{\rho + \gamma K} + i \right), \tag{4.12}$$

and

$$F(X, K) = \frac{K}{\beta_1 - 1} \left(\frac{\varphi K}{\rho} + \frac{c}{\rho + \gamma K} + i \right) \left(\frac{X}{\overline{X}(K)} \right)^{\beta_1}. \tag{4.13}$$

From equation (4.12) the next proposition immediately follows.

Proposition 4.2 *When the firm can choose the timing but not the scale of the investment, the investment trigger $\overline{X}(K)$ is decreasing in γ.*

When the scale of the project is fixed, a more intense learning process accelerates the investment. This result is also intuitive. For a given capacity, a larger γ implies a higher value of the project so that a lower X is needed to induce the firm to invest.

4.3.2 Joint determination of timing and scale

Up to this point, we considered the timing and scale dimensions separately. We showed that when the scale of the investment is flexible but the timing is not, a more intense learning process implies that a firm should invest in a larger capacity. Also, we showed that when the timing is flexible but the scale is fixed, the learning curve accelerates the investment. These findings may suggest that investment should occur early and on a large scale. However, timing and scale of investment involve a trade-off. The key observation is that a large scale investment is costly when investment is early and the market is still small. On the other hand, later investment, i.e. investing at a moment after the market has grown large, can sustain an increased scale. Here, we bring the timing and scale dimensions together and see how, in the presence of the leaning curve, the timing-scale trade-off is optimally resolved.

Given the timing rule defined by (4.12), the firm determines the project scale to maximize the value of its investment option. That is, the optimal capacity $\overline{K}$ maximizes (4.13). Rearranging the first order condition, $F_K(X, \overline{K}) = 0$, yields

$$(\beta_1 - 2)\,\frac{\varphi\overline{K}}{c\rho} - \frac{\rho + \beta_1\gamma\overline{K}}{\left(\rho + \gamma\overline{K}\right)^2} = \frac{i}{c}, \tag{4.14}$$

which implicitly defines $\overline{K}$. It is easy to verify that a finite positive solution for $\overline{K}$ exists if $\beta_1 > 2$. Substituting the optimal capacity $\overline{K}$ in (4.12) gives the value for the optimal investment threshold $\overline{X}(\overline{K})$, also denoted simply by $\overline{X}$.

A closed form expression for the optimal capacity $\overline{K}$, and therefore for the investment trigger $\overline{X}$, is not available. Yet, in the Appendix we show that, when it exists, $\overline{K}$ is uniquely determined by (4.14). Furthermore, we show the following analytical results regarding the effects of the steepness γ of the learning curve on the scale and timing of investment.

Proposition 4.3 *If the firm can choose the timing and scale of learning investment, the following holds. The optimal scale $\overline{K}$ is an inverse-U-shaped function of γ. The investment trigger $\overline{X}$ is decreasing in γ.*

Given the timing-scale trade-off, the firm faces two alternatives. The first is to benefit from a large capacity at the cost of delaying investment and entry into the market. The second is to invest and to earn profits early but at the cost of setting a small scale production facility. Proposition 3 implies that the first alternative is preferred if γ is low while the second is chosen if γ is high. We interpret our findings as follows.

When γ is low, the learning effect, although present, is weak and the firm needs a high per-period production rate to reduce the marginal cost at a sufficient speed. For this reason, the benefits of a larger capacity outweigh the costs of a delayed investment and optimality implies that investment should occur relatively late but on a large scale. On the contrary, when γ is high, the learning process is fast even with a low production rate. Therefore, the benefits of an additional unit of capital, in terms of an increased speed of learning, are small compared to the cost of delaying

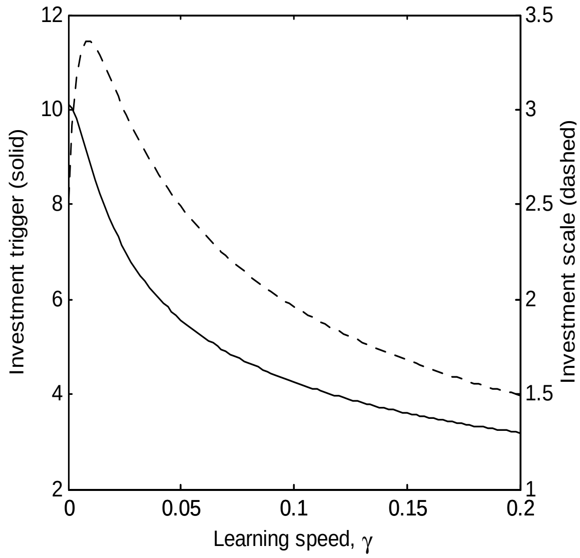

Figure 4.1: Optimal scale and investment trigger. Parameter values are: $c = 5$, $\rho = 0.06$, $\mu = 0$, $\sigma = 0.1$, $\varphi = 1$, and $i = 1$.

the investment. For this reason, the optimal strategy is to reduce the scale of the project and to enter the market early.

Proposition 3 says that a more intense learning process accelerates investment, i.e. faster learning means a lower investment trigger. This result is not obvious, because there are two opposing forces at work. The total effect of γ on the investment trigger is given by $d\overline{X}(\overline{K})/d\gamma = \partial\overline{X}(\overline{K})/\partial\gamma + \partial\overline{X}(\overline{K})/\partial\overline{K}\ d\overline{K}/d\gamma$. The first term, always negative, is the direct effect of γ on the investment trigger. The second term is the indirect effect of γ through its influence on the optimal capacity $\overline{K}$. The sign of this term is ambiguous and is positive when $\overline{K}$ increases in γ. Here, the timing-scale trade-off becomes clearer. Potentially, the downward effect of the learning curve on the investment trigger, meaning earlier investment, may be more than compensated by the upward effect due to the larger capacity. In this setup the first effect is always dominant, however.

Summarizing, our findings suggest that the following rule of thumb effectively describes the investment strategy in the presence of the learning curve. If the learning process is slow, invest relatively late and on a larger scale. If the learning process is fast, invest early and on a smaller scale.

To demonstrate the quantitative effect of learning on investment, we provide a numerical example. Parameter values are set as follows. The initial marginal cost is $c = 5$, the discount rate is equal to $\rho = 0.06$, the drift parameter is set equal to $\mu = 0$, and $\varphi = 1$. Finally, the cost of one unit of capital is $i = 1$.

Figure 4.1 presents the effects of γ on scale and timing of investment. The dashed curve in Figure 4.1 plots $\overline{K}$ as a function of γ. As confirmed in Proposition 3, the optimal capacity $\overline{K}$ is an inverse-U-shaped function. The steep increase of $\overline{K}$ for low γ has a substantial effect: the scale of investment is larger with learning than without learning up to γ equal to about 0.05. The solid curve shows that $\overline{X}$ is a monotonic and decreasing function of γ. Both timing and scale can be greatly affected by the learning curve. For small learning rates, investment is late and on a large scale. For larger learning rates, say for values of γ above 0.05, investment is taken early and on a relatively small scale.

4.4 Initial losses

In the standard real options analysis, firms invest above the break-even point, i.e. they make (substantial) profits from the moment of investment on. However, learning investment brings additional long-term incentives, which may tempt the firm to accept some initial losses. This section, therefore, tries to answer the following questions. Do firms accept some initial losses to benefit from the learning effects? Provided the answer is positive, how large are these losses compared to the initial investment cost? Is steeper learning related to higher initial losses?

Profits or losses at the moment of investment are equal to $(\overline{X} - \varphi\overline{K} - c)\overline{K}$. It is easy to verify that initially firms make losses when investing in learning technologies. For our baseline parameter values, this is demonstrated in Figure 4.2.A. The firm invests at losses already at such low values of γ as 0.015, which implies that the long-run learning incentives are already strong even when the learning curve is relatively flat. Interestingly, after some point the level of losses starts to decrease in γ. This can be explained by the decreasing scale of learning investment in γ (see Figure 4.1). Very steep learning implies that the firm invests in small capacity, so losses right after the investment time do not have to be large to support sufficient learning.

Next, we focus on cumulative losses up to the time when the production breaks even. Let us denote by L the expected present value of the stream of losses being incurred before the first time that per-period profits are zero. This will measure how much more capital the firm needs to furnish beyond the initial investment cost.

To simplify notation, at the moment of investment, reset time to $t = 0$. Given the profit flow

$$\pi_t = \left(X_t - \varphi\overline{K} - ce^{-\gamma Q_t}\right)\overline{K} = (X_t - \varphi\overline{K} - ce^{-\gamma\overline{K}t})\overline{K},$$

the break-even point is achieved at the stopping time $\tau_{BE} = \inf\{t \geq 0 : \pi_t = 0\} = \inf\{t \geq 0 : X_t = \varphi\overline{K} + ce^{-\gamma\overline{K}t}\}$. The break-even point can be denoted as a time-dependent threshold on X, namely $X_{BE}(t) = \varphi\overline{K} + ce^{-\gamma\overline{K}t}$. Note that π_t can be written as $\pi(X, t)$.

The value of the expected stream of losses from point (X, t) up to the break-even point is denoted by $L(X, t)$. At the moment of the investment it is given by

$$L(\overline{X}, 0) = E\int_0^{\tau_{BE}} \pi(X_t, t)e^{-\rho t}dt.$$

It follows from standard arguments that L must satisfy the partial differential equation

$$\rho L = \alpha X L_X + \frac{1}{2}\sigma^2 X^2 L_{XX} + L_t + \pi(X, t),$$

It is solved subject to a boundary condition at $X_{BE}(t)$ where L should be equal to zero:

$$L(X_{BE}(t), t) = 0. \tag{4.15}$$

This needs to be solved numerically; we apply the finite-difference method.[7]

[7] Apart from (4.15), we need other boundary conditions in the (X, t) space. At the absorbing state $X = 0$, we have that

$$L(0, t) = -\left(\frac{\varphi\overline{K}}{\rho} + \frac{ce^{-\gamma\overline{K}t}}{\rho + \gamma\overline{K}}\right)\overline{K}.$$

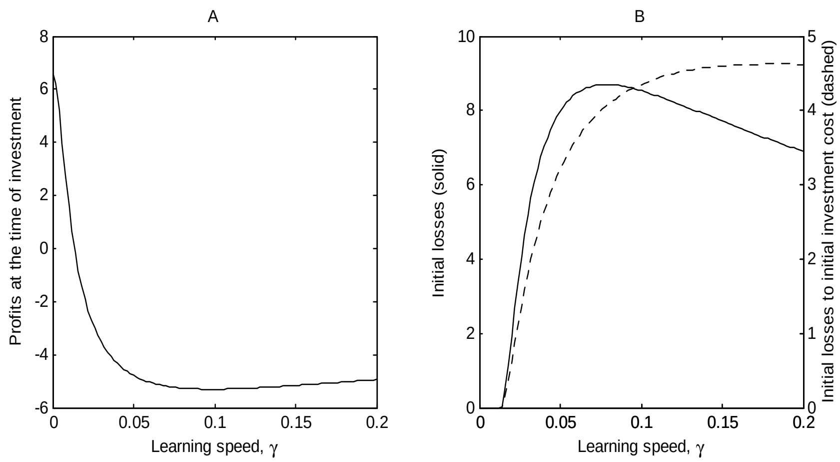

Figure 4.2: Profits at the time of investment and the present value of initial losses as a function of learning speed γ. Parameter values are: $c = 5$, $\rho = 0.06$, $\mu = 0$, $\sigma = 0.1$, $\varphi = 1$, and $i = 1$.

The results for the baseline parameter values are presented in Figure 4.2.B. The solid curve plots the present value of initial cumulative losses. This value can exceed 8.50, which happens for values of the learning rate γ around 0.08. Initial cumulative losses then decrease for higher γ. This pattern originates from the non-monotonicity of instantaneous losses presented in Figure 4.2.A. However, the non-monotonic shape is much steeper here. This is because with faster learning, the break-even point is reached sooner and the cumulative initial losses decrease. This implies that in terms of required financial slack, investment in technologies with moderate learning effects is most demanding.

It is interesting to compare the initial cumulative losses to the initial

For t high enough, learning is almost completed so profits become independent of cumulative output, and thus L is time-independent. The break-even point is then $X_{BE} = \varphi\overline{K}$. $L(X)$ must satisfy the ODE, $\rho L = \alpha X L_X + \frac{1}{2}\sigma^2 X^2 L_{XX} + (X - \varphi\overline{K})\overline{K}$, subject to $L(X_{BE}) = 0$. This implies that

$$L(X) = \left(\frac{X}{\rho - \alpha} - \frac{\varphi\overline{K}}{\rho}\right)\overline{K} - \left(\frac{X}{\varphi\overline{K}}\right)^{\beta_1}\left(\frac{\varphi\overline{K}}{\rho - \alpha} - \frac{\varphi\overline{K}}{\rho}\right)\overline{K}.$$

cost of investment. The dashed curve in Figure 4.2.B plots the ratio of these variables for different values of γ. It demonstrates that initial cumulative losses can be very substantial and well exceed the initial cost of investment. In the baseline case the ratio exceeds four and is relatively flat for γ sufficiently large. (The flat shape is caused by the decreased scale of investment for high γ.)

4.5 Downside risk

The analysis so far assumed that the only source of uncertainty is the diffusion risk in the market demand. However, many investments and technologies may be susceptible to downside jump risk. This may be particularly relevant for learning investment. Frontier technologies in such investments can be superseded by even newer technologies or may prove to be technologically flawed.

To examine the effects of downside risk on learning investment, we introduce the possibility that all along the time period after the investment, with probability λdt an event can occur that results in the death of the project, where λ is a positive constant. The analysis follows the same steps as those presented in Section 4.3. Therefore, only the key steps are highlighted here.

The value of the production facility in place is equal to the discounted stream of profits, so that

$$V(X,K,Q) = \frac{XK}{\rho-\mu+\lambda} - \frac{\varphi K^2}{\rho+\lambda} - \frac{ce^{-\gamma Q}K}{\rho+\gamma K+\lambda}.$$

Note that the difference with expression (4.6) is that the discount rate is augmented with the expiry rate λ. For a given capacity K, the investment is optimally undertaken when X reaches the upper trigger $\overline{X}(K)$ given by

$$\overline{X}(K) = \frac{\beta_1(\rho-\mu+\lambda)}{\beta_1-1}\left(\frac{\varphi K}{\rho+\lambda} + \frac{c}{\rho+\gamma K+\lambda} + i\right).$$

The optimal scale $\overline{K}$ maximizes the value of the option to invest and is implicitly given by

$$(\beta_1-2)\frac{\varphi\overline{K}}{c(\rho+\lambda)} - \frac{\rho+\beta_1\gamma\overline{K}+\lambda}{\left(\rho+\gamma\overline{K}+\lambda\right)^2} = \frac{i}{c}.$$

Then the entry trigger equals $\overline{X} = \overline{X}(\overline{K})$.

It is straightforward to derive that both $\overline{X}$ and $\overline{K}$ increase in λ. To verify whether λ can have quantitatively different effects on learning and non-learning investment, we use the baseline parameter values with different learning rates and a range of small realistic values for λ between 0 and 0.1. Figure 4.3 presents the results. The solid curve plots the values for investment with no learning effects ($\gamma = 0$), the dashed curve represents intermediate learning effects ($\gamma = 0.1$), and the dotted curve represents a steep learning curve ($\gamma = 0.2$).

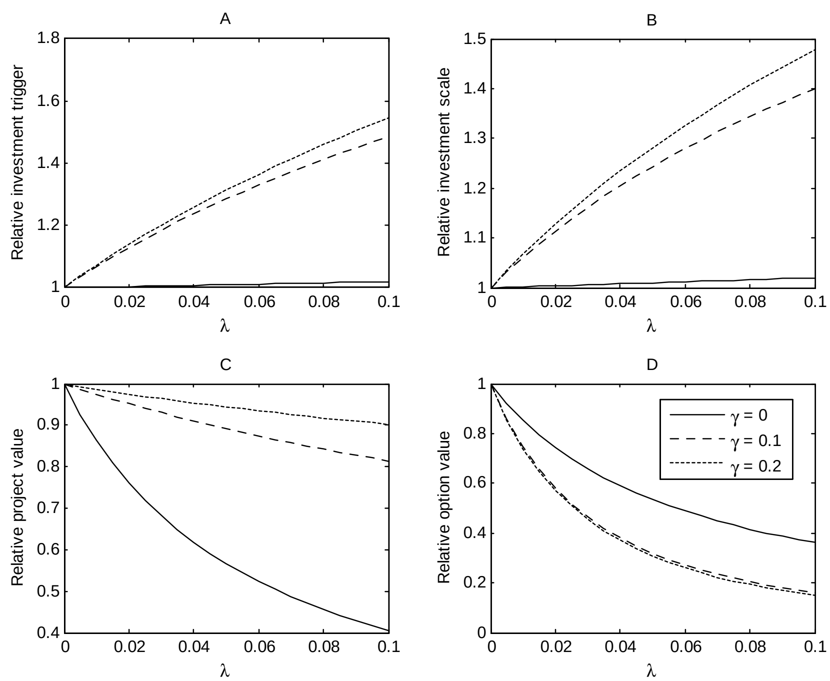

Figure 4.3: The effects of downside risk at rate λ. The solid curve plots $\gamma = 0$, the dashed curve plots $\gamma = 0.1$ and the dotted curve plots $\gamma = 0.2$. Other parameter values are: $c = 5$, $\rho = 0.06$, $\mu = 0$, $\sigma = 0.1$, $\varphi = 1$, $i = 1$, and $X_0 = 2$.

Figures 4.3.A and 4.3.B show that investment without learning is rather insensitive to downside risk. The opposite is observed for learning investment. Already for intermediate learning ($\gamma = 0.1$), both timing and scale are very sensitive to the presence of downside risk, even when this risk is small. In fact, the effects for the very steep learning curve ($\gamma = 0.2$) are not much stronger any more. These observations can be explained by considering the amount of initial losses associated with learning investments. If investment generates losses in early stages, then an early expiry before any long-term gains are realized is particularly costly. Because the losses are the largest for intermediate learning curves, these investments are relatively most sensitive.

Next, we look at the effects on the value of investment in place and on the value of the option to invest. Figure 4.3.C plots the ratio of the value of investment in place for a range of λ's to the value of investment in place for $\lambda = 0$. It shows how much downside risk decreases the value of the production facility at the moment of investment. The figure indicates that the smaller the learning effects, the more value is lost at the moment of investment with the introduction of downside risk. This means that learning investment has more leeway in adjusting timing and scale so that the value at the moment of investment is not so much affected. However, this flexibility comes at a cost. Figure 4.3.D plots similar ratios but of the value of the option to invest to show how much the investment option value is destroyed by downside risk. Figure 4.3.D takes the distortion of the terms of investment into account, which is not reflected in Figure 4.3.C. In this case, the steeper the learning curve, the more value is lost due to downside risk. This is because learning investments are substantially delayed, which helps to maintain the value of the project at the investment time (Figure 4.3.C), but weakens the learning potential and destroys the value of the option to invest (Figure 4.3.D). It is worth noting that the difference between the two learning cases with $\gamma = 0.1$ and $\gamma = 0.2$ are very small, which shows again that investments in technologies with moderate learning effects are relatively most vulnerable.

4.6 Conclusions

This work investigates optimal investment behavior in a situation where production costs decrease over time due to learning. The firm under consideration operates in an uncertain output market with fluctuating demand. To maximize the effects of the learning curve, the firm would invest early and on a large scale. However, such a policy is costly in the short-run and risky. We show that, if the learning curve is flat, firms optimally invest late and on a large scale. On the other hand, when the learning curve is steep, early and small scale investments are optimal.

We further show that learning investment is associated with large expected losses in the early stages after the firm undertook the investment. The reason is that, due to learning, production costs reduce over time, implying that in the beginning they are still high. The implication of the occurrence of initial losses is that learning investment is very sensitive to downside risk in the sense that an event can occur leading to the end of the project at a time too early for learning to have caused sufficient cost reductions. The expected losses and distortions are particularly strong for investment with moderate learning curves.

Despite its focus on optimal firm decisions, our analysis has some clear policy implications. Investing in technologies with very steep learning curves, like many information technologies, can be efficiently undertaken by firms. However, investing in technologies with moderate learning curves is more difficult and more financially demanding for firms. Frictions, for example financing constraints, may easily lead to suboptimal investments in these technologies. If there are some positive externalities of technological investments, such as learning spillovers or positive environmental effects, then suboptimal investment in such technologies may be especially costly from a welfare perspective. This category of technologies, i.e. with moderate learning and large externalities, may include some energy technologies. In this case, public support of investment, e.g., in the form of guaranties, may be warranted.

4.A Appendix: Proofs

4.A.1 Proof of Proposition 4.1

Totally differentiating condition (4.6) and rearranging yields

$$d\overline{K}/d\gamma = -V_{K_\gamma}(X, \overline{K}, 0)/V_{KK}(X, \overline{K}, 0).$$

From (4.6),

$$V_{K\gamma}(X, \overline{K}, 0) = 2\rho\gamma c\overline{K}/\left(\rho + \gamma\overline{K}\right)^3 > 0$$

and

$$V_{KK}(X, K, 0) = -2\varphi/\rho - \gamma^2\overline{K}c/\left(\gamma\overline{K} + \rho\right)^3 < 0.$$

Hence, the effect of γ on the optimal capacity, $d\overline{K}/d\gamma$, is positive. ■

4.A.2 Proof of Proposition 4.2

In the text. ■

4.A.3 Optimal capacity: existence and uniqueness

Differentiating (4.13) with respect to K yields the first order condition

$$\begin{aligned} F_K(X, \overline{K}) &= \frac{1}{\beta_1 - 1}\left(\frac{X}{\overline{X}(K)}\right)^{\beta_1}\left\{\frac{\varphi\overline{K}}{\rho} + \frac{c}{\gamma\overline{K} + \rho} + i - \right. \\ &\left. (\beta_1 - 1)\left[\frac{\varphi\overline{K}}{\rho} - \frac{\gamma\overline{K}c}{\left(\rho + \gamma\overline{K}\right)^2}\right]\right\} \\ &= 0. \end{aligned} \tag{4.16}$$

Rearranging the term between the curly brackets yields condition (4.14). Existence of a positive finite solution to (4.14) was argued in the text and requires that $\beta_1 > 2$.

The second order condition for maxima at the stationary points $\overline{K}$ is

$$\begin{aligned} F_{KK}(X, \overline{K}) &= \frac{1}{\beta_1 - 1}\left(\frac{X}{\overline{X}(K)}\right)^{\beta_1}\left\{(2 - \beta_1)\left[\frac{\varphi}{\rho} - \frac{\gamma c}{\left(\gamma\overline{K} + \rho\right)^2}\right] + \right. \\ &\left. (1 - \beta_1)\frac{\gamma^2\overline{K}c}{\left(\gamma\overline{K} + \rho\right)^3}\right\} \\ &< 0. \end{aligned} \tag{4.17}$$

To verify that the inequality holds, we use the first order condition. Note that, because the first three terms between the curly brackets of (4.16) are positive and $\beta_1 > 1$, a necessary requirement for condition (4.16) to hold is

$$\frac{\varphi\overline{K}}{\rho} - \frac{\gamma\overline{K}c}{\left(\gamma\overline{K}+\rho\right)^2} > 0. \tag{4.18}$$

Using (4.18) and $\beta_1 > 2$ in the left hand side of (4.17), we confirm the inequality in (4.17). Given that $F(X,K)$ is a continuous and smooth function, local concavity at the stationary points implies that the stationary point is unique. Hence, $\overline{K}$ (when it exists) is unique and maximizes $F(X,K)$. ■

4.A.4 Proof of Proposition 4.3

The effect of γ on the optimal capacity is given by

$$d\overline{K}/d\gamma = -F_{K\gamma}(X,\overline{K})/F_{KK}(X,\overline{K}).$$

Differentiating and rearranging the first order condition (4.16) with respect to γ yields

$$F_{K\gamma}(X,\overline{K}) = \frac{1}{\beta_1-1}\left(\frac{X}{\overline{X}(K)}\right)^{\beta_1}\frac{c\overline{K}}{\left(\rho+\gamma\overline{K}\right)^3}\left[\rho\left(\beta_1-2\right)-\beta_1\gamma\overline{K}\right]. \tag{4.19}$$

Recalling that $F_{KK}(X,\overline{K}) < 0$, the optimal capacity increases in γ when $F_{K\gamma}(X,\overline{K}) > 0$ and decreases otherwise. The sign of $F_{K\gamma}(X,\overline{K})$ is identical to the sign of the term between the square brackets. Define $\Gamma(\gamma) = \rho\left(\beta_1-2\right) - \beta_1\gamma\overline{K}$. The optimal capacity $\overline{K}$ is an inverse-U-shaped function of γ as claimed in the proposition if $\Gamma(\gamma) > 0$ for low values of γ, $\Gamma(\gamma) < 0$ for high values of γ and if there is only one value of γ which satisfies $\Gamma(\gamma) = 0$.

Assume that (4.14) holds and that $\beta_2 > 2$ so that a finite positive solution for $\overline{K}$ exists. When γ is small, given that $\lim_{\gamma\to 0}\overline{K}$ is bounded, $\Gamma(\gamma)$ is positive. Assume, now, that $\overline{K}$ is always increasing in γ. This assumption requires that $\Gamma(\gamma) > 0$ for every combination of γ and $\overline{K}$. But given that $\rho\left(\beta_1-2\right)$ is constant, for sufficiently large values of γ, it must be that $\Gamma(\gamma) < 0$, contradicting the initial hypothesis. Hence, for large values of γ there exists a region where $d\overline{K}/d\gamma < 0$, i.e. $\Gamma(\gamma) < 0$.

Finally, we show that $\Gamma(\gamma)$ changes its sign only once with increasing γ. Differentiating $\Gamma(\gamma)$ with respect to γ yields

$$\Gamma'(\gamma) = -\beta_1 \overline{K} - \beta_1 \gamma d\overline{K}/d\gamma, \tag{4.20}$$

which is negative if $d\overline{K}/d\gamma \geq 0$, that is if $\Gamma(\gamma) \geq 0$. It follows that there exists only one value of γ that satisfies $\Gamma(\gamma) = 0$.

The total effect of γ on the optimal investment trigger $\overline{X}$ is given by

$$\frac{d\overline{X}}{d\gamma} = \frac{\partial \overline{X}(\overline{K})}{\partial \gamma} + \frac{\partial \overline{X}(\overline{K})}{\partial \overline{K}} \frac{d\overline{K}}{d\gamma} = \frac{\partial \overline{X}(\overline{K})}{\partial \gamma} - \frac{\partial \overline{X}(\overline{K})}{\partial \overline{K}} \frac{F_{K_\gamma}(X, \overline{K})}{F_{KK}(X, \overline{K})}. \tag{4.21}$$

Rearranging, we obtain

$$\frac{d\overline{X}}{d\gamma} = -\frac{(\rho - \mu)\, 2\beta_1 c \varphi \gamma \overline{K}^2}{(\beta_1 - 2)\left[\varphi\left(\gamma\overline{K} + \rho\right)^3 - c\gamma\rho^2\right] + \beta_1 c \gamma^2 \rho \overline{K}}. \tag{4.22}$$

Condition $\frac{\varphi}{\rho} - \frac{\gamma c}{\left(\rho + \gamma\overline{K}\right)^2} > 0$ implies that the term between the square brackets in the denominator is always positive. This, together with $\beta_2 > 2$ and $\rho > \mu$, yields that $d\overline{X}/d\gamma < 0$. ■

Bibliography

Abbring, J., and J.R. Campbell, 2010, Last-in first-out oligopoly dynamics, *Econometrica*, Vol. 78, 1491-1527.

Abel, A.B., 1983, Optimal investment under uncertainty, *American Economic Review*, Vol. 73, 228-33.

Abel, A.B., 1984, The effects of uncertainty on investment and the expected long-run capital stock, *Journal of Economic Dynamics and Control*, Vol. 7, 39-54.

Abel, A.B., 1985, A stochastic model of investment, marginal q and the market value of the firm, *International Economic Review*, Vol. 26, 305-22.

Acharya, V., H. Almeida , and M. Campello, 2007, Is cash negative debt? A hedging perspective on corporate financial policies, J*ournal of Financial Intermediation*, Vol. 16, 515–554.

Almeida, H., M. Campello, and M.S. Weisbach, 2004, The cash flow sensitivity of cash, *Journal of Finance*, Vol. 59, 1777–1804.

Aguerrevere, F., 2009, Real options, product market competition and asset returns, *Journal of Finance*, Vol. 64, 957–983.

Anderson, S.T., D. Friedman, and R. Oprea, 2010, Preemption games: Theory and evidence, *American Economic Review*, Vol. 100, 1778–1803.

Argote, L., S. Beckham, and D. Epple, 1990, The persistence and transfer of learning in industrial settings, *Management Science*, Vol. 36, 140–154.

Auerswald, P., S. Kauffman, J. Lobo, K. Shell, 2000, The production recipes approach to modeling technological innovation: An application to learning by doing, *Journal of Economic Dynamics and Control*, Vol. 24, 389-450.

Bahk, B.H., and M. Gort, 1993, Decomposing learning by doing in new plants, *Journal of Political Economy*, Vol. 101, 561–583.

Bar-Ilan, A., and W.C. Strange, 1999, The timing and intensity of investment, *Journal of Macroeconomics*, Vol. 21, 57–77.

Barclay M.J., E. Morellec, and C.W. Smith, 2006, On the debt capacity of growth options, *Journal of Business*, Vol. 1, 37-59.

Bates, T.W., K.M. Kahle, and R.M. Stulz, 2009, Why do U.S. firms hold so much more cash than they used to?, *Journal of Finance*, Vol. 64, 1985-2021.

Berk, R., C. Green, and V. Naik , 1999, Valuation and return dynamics of new ventures, *Journal of Finance*, Vol. 54, 1553–1607.

Bernanke, B.S., 1983, Irreversibility, uncertainty, and cyclical investment,*Quarterly Journal of Economics*, Vol. 98, 85-106.

Besanko, D., U. Doraszelski, Y. Kryukov, and M. Satterthwaite, 2010, Learning by doing, organizational forgetting, and industry dynamics, *Econometrica*, Vol. 78, 453–508.

Boone J., 2009, A new way to measure competition, *Economic Journal*, Vol. 118, 1245–1261.

Boone, J., H. van der Wiel, and J. van Ours, 2007, How (not) to measure competition, CEPR Discussion Papers 6275.

Boyle, G.W., and Graeme A. Guthrie, 2003, Investment, uncertainty, and liquidity, *Journal of Finance*, Vol. 58, 2143–2166.

Brady, B., and R.J. Bos, 2002, *Record Defaults in 2001: The Result of Poor Credit Quality and a Weak Economy*, Standard & Poor's, New York.

Brueckner, J.K., N. Raymon, 1983, Optimal production with learning by doing, *Journal of Economic Dynamics and Control*, Vol. 6, 127-135.

Caballero, R.J., 1991, On the sign of the investment-uncertainty relationship, *American Economic Review*, Vol. 81, 279-288.

Cabral, L.M.B, and M.H. Riordan, 1994, The learning curve, market dominance, and predatory pricing, *Econometrica*, Vol. 62 1115–1140.

Cabral, L.M.B, and M.H. Riordan, 1997, The learning curve, predation, antitrust, and welfare, *Journal of Industrial Economics*, Vol. 45, 155–169.

Campbell, J.Y., M. Lettau, B. Malkiel, and Y. Xu, 2001, Have individual stocks become more volatile? An empirical exploration of idiosyncratic risk, *Journal of Finance*, Vol. 56, 1-43.

Campello, M., 2003, Capital structure and product market interactions: Evidence from the business cycles, J*ournal of Financial Economics*, Vol. 68, 353-378.

Capozza, D.R., and L. Yuming, 1994, The intensity and timing of investment: The case of land, *American Economic Review*, Vol. 84, 889-904.

Carlson, M., A. Fisher, and R. Giammarino, 2004, Corporate investment and asset price dynamics: Implications for the cross-section of returns, *Journal of Finance*, Vol. 59, 2577-2603.

Chaney, T., X. Gabaix, and T. Philippon, 2002, Firm volatility, *Working Paper.*

Chen, Z., and L. Epstein, 2002, Ambiguity, risk and asset returns in continuous time, *Econometrica*, Vol. 70, 1403-1443.

Comin, D., and T. Philippon, 2005, The rise in firm-level volatility: Causes and consequences, *NBER Working Paper n.11388.*

Dangl, T., 1999, Investment and capacity choice under uncertain demand, *European Journal of Operational Research*, Vol 117, 415-428.

Dasgupta, P., and J. Stiglitz, 1988, Learning-by-doing, market structure and industrial and trade policies, *Oxford Economic Papers*, Vol. 40, 246–268.

Dixit, A.K., 1993, *The Art of Smooth Pasting*, Harwood Academic Publisher, Chur, Switzerland.

Dixit, A.K., 1989, Entry and exit decisions under uncertainty, *Journal of Political Economy*, Vol. 97, 620-638.

Dixit, A.K. and R.S. Pindyck, 1994, *Investment Under Uncertainty*, Princeton University Press, Princeton, NJ.

Dixit, A.K. and J.E. Stiglitz, 1977, Monopolistic competition and optimum product diversity, *American Economic Review*, Vol. 67, 297-308.

Dunne, T., M.J. Roberts, and L. Samuelson, 1988, Patterns of firm entry and exit in U.S. manufacturing industries, *Rand Journal of Economics*, Vol. 19, 495–515.

Dutta, J., and K. Prasad, 1996, Learning by observation within the firm, *Journal of Economic Dynamics and Control*, Vol. 20,1395-1425.

Ellsberg, D., 1961, Risk, ambiguity, and the Savage axiom, *Quarterly Journal of Economics,* Vol. 75, 643-669.

Epstein, L.G., J. Noor, and A.Sandroni , 2010, Non-Bayesian learning, *B.E. Journal of Theoretical Economics (Advances)*, Vol. 10, Iss. 1, Article 3.

Epstein, L.G. and M. Schneider, 2010, Ambiguity and asset markets, NBER Working Paper 16181.

Ericson, R., and A. Pakes, 1995, Markov-perfect industry dynamics: A framework for empirical work, *Review of Economic Studies*, Vol. 62, 53–82.

Faulkender, M., and R. Wang, 2006, Corporate financial policy and the value of cash, *Journal of Finance*, Vol. 61, 1957–1990.

Fischbacher, U., 2007, z-Tree: Zurich Toolbox for ready-made economic experiments, *Experimental Economics*, Vol. 10, 171-178.

Fox, C.R., and A. Tversky, 1995, Ambiguity aversion and comparative ignorance, *Quarterly Journal of Economics,* Vol. 110, 585-603.

Frésard, L., 2010, Financial strength and product market behavior: The real effects of corporate cash holdings, *Journal of Finance*, Vol. 65, 1097-1122.

Frésard, L., and P. Valta, 2011, Competitive pressure and corporate policies, *Working Paper.*

Fudenberg, D., and J. Tirole, 1983, Learning-by-doing and market performance, *The Bell Journal of Economics*, Vol. 14, 522–530.

Gaspar, J.M., and M. Massa, 2006, Idiosyncratic volatility and product market competition, *Journal of Business*, Vol. 79, 3125–3152.

Ghirardato, P., F. Maccheroni, and M. Marinacci, 2004, Differentiating ambiguity and ambiguity attitude, *Journal of Economic Theory*, Vol 118,133–173.

Ghirardato, P., F. Maccheroni, and M. Marinacci, 2008, *Revealed Ambiguity and its Consequences: Updating, in Advances in Decision Making under Risk and Uncertainty* (M. Abdellaoui and J. D. Hey, eds), Springer, 2008.

Gilboa, I., and D. Schmeidler, 1989, Maximin expected utility with non-unique prior, *Journal of Mathematical Economics*, Vol. 18, 141-153.

Gilboa, I., and D. Schmeidler, 1993, Updating ambiguous beliefs, *Journal of Economic Theory*, Vol. 59, 33-49.

Gruber, H., 1992, The learning curve in the production of semiconductor memory chips, *Applied Economics*, Vol. 24, 885–894.

Gryglewicz, S., 2011, A theory of corporate financial decisions with liquidity and solvency concerns, *Journal of Financial Economics*, Vol. 99, 365-384.

Guiso, L., and G. Parigi, 1996, Investment and demand uncertainty, *Quarterly Journal of Economics*, Vol. 114, 185-227.

Hartman, R., 1972, The effects of price and cost uncertainty on investment, *Journal of Economic Theory*, Vol. 5, 258-66.

Hirsch, W., 1952, Manufacturing progress functions, *Review of Economics and Statistics*, Vol. 34, 143–155.

Hopenhayn, H.A., 1992, Entry, exit, and firm dynamics in long-run equilibrium, *Econometrica, Vol.* 60, 1127–1150.

Huisman, K.J.M., and P.M. Kort, 2003, Strategic investment in technological innovations, *European Journal of Operational Research*, Vol. 144, 209-223.

Irivine, P.J., and J. Pontiff, 2009, Idiosyncratic return volatility, cash flows, and product market competition, *Review of Financial Studies*, Vol. 22, 1149-1177.

Ivanov, A., 2011, Attitudes to ambiguity in one-shot normal-form games: An experimental study, *Games and Economic Behavior*, Vol. 71, 366-394.

Jaffray, J.Y. (1994), *Dynamic decision making with belief functions*, in Yager et al. (ed.), Advances in the Dempster-Shafer Theory of Evidence, Wiley.

Jensen, M.C., 1986, Agency costs of free cash flow, corporate finance and takeovers, *American Economic Review*, Vol. 76, 323-329.

Jensen, M.C., and W.H. Meckling, 1976, Theory of the firm: Managerial behavior, agency costs and ownership structure, *Journal of Financial Economics*, Vol. 3, 305-36

Kaplan, E., and P. Meier, 1958, Nonparametric estimation from incomplete observations, *Journal of the American Statistical Association*, Vol. 53, 457–481.

Kellog, R., 2010, The effect of uncertainty on investment. evidence from Texas oil drilling, NBER Working Paper 16541.

Keppo, J., Smith L., and D. Davydov, 2008, Optimal electoral timing: Exercise wisely and you may live longer, *Review of Economic Studies*, Vol. 75, 597–628.0.

Lambrecht, B.M., 2004, The timing and terms of mergers motivated by economies of scale, *Journal of Financial Economics*, Vol. 72, 41-62.

Leland, H.E., 1994, Corporate debt value, debt covenants, and optimal capital structure, *Journal of Finance*, Vol. 49, 987-1019.

Leland, H.E., and K.B. Toft, 1996, Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads, *Journal of Finance*, Vol. 51, 987–1019.

Lieberman, M., 1984, The learning curve and pricing in the chemical processing industries, *RAND Journal of Economics*, Vol. 15, 213–228.

Lindenberg, E.B., and S.A. Ross, 1981, Tobin's q ratio and industrial organization, *Journal of Business*, Vol. 54, 1–32.

Lins, K.V., H. Servaes, and P. Tufano, 2010, What drives corporate liquidity? An international survey of cash holdings and lines of credit, *Journal of Financial Economics*, Vol. 98, 160–176.

List, J.A., and M.S. Haigh, 2010, Investment under uncertainty: Testing the options model with professional traders, *Review of Economics and Statistics*, Vol. 92, 974–984.

MacDonald, R., and D. Siegel, 1986, The value of waiting to invest, *Quarterly Journal of Economics*, Vol. 101, 707-28.

Majd, S., and R.S. Pindyck, 1989, The learning curve and optimal production under uncertainty, *RAND Economic Journal*, Vol. 20, 331–343.

Melitz, M.J., 2003, The impact of trade on intra-industry reallocations and aggregate industry productivity, *Econometrica*, Vol. 71, 1695-1725.

Mello, A.S., and J.E. Parsons, 2000, Hedging and liquidity, *Review of Financial Studies*, Vol. 13, 127–153.

Miao, J., 2005, Optimal capital structure and industry dynamics, *Journal of Finance*, Vol. 13, 127–153, 2621–2659.

Miao, J., and N. Wang, 2007, Investment, consumption, and hedging under incomplete markets, *Journal of Financial Economincs*, Vol. 86, 608-642.

Miao, J., and N. Wang, 2011, Risk, uncertainty, and option exercise, *Journal of Economic Dynamics and Control*, Vol. 35, 442-461.

Moel, A., and Tufano, P., 2002, When are real options exercised? An empirical study of mine closings, *Review of Financial Studies*, Vol. 15, 35–64.

Morellec, E., and B. Nikolov, 2011, Cash holdings and competition, *Working Paper*.

Morellec, E., and A. Zhdanov, 2005, The dynamics of mergers and acquisitions, *Journal of Financial Economics*, Vol. 77, 649-672.

Murto, P., and M. Terviö, 2009, Exit options and dividend policy under liquidity constraints, *Working Paper*.

Nishimura, K.G., and H. Ozaki, 2004, Search and knightian uncertainty, *Journal of Economic Theory*, Vol. 119, 299–333.

Nishimura, K.G., and H. Ozaki, 2007, Irreversible investment and knightian uncertainty, *Journal of Economic Theory*, Vol. 136, 668-694.

Novy-Marx, R., 2007, An equilibrium model of investment under uncertainty, *Review of Financial Studies*, Vol. 5, 1461–1502.

Oprea, R., D. Friedman, and S.T. Anderson, 2009, Learning to wait: A laboratory experiment, *Review of Economic Studies*, Vol. 76, 1103-1124.

Opler, T., L. Pinkowitz, R. Stulz, and R. Williamson, 1999, The determinants and implications of corporate cash holdings, *Journal of Financial Economics*, Vol. 52, 3–46.

Phillips, G.M., 1995, Increased debt and industry product markets: An empirical analysis, *Journal of Financial Economics*, Vol. 37, 189–238.

Pires, C.P., 2002, A rule for updating ambiguous beliefs, *Theory and Decision,* Vol. 53, 137-152.

Polborn, M., 2006., Investment under uncertainty in dynamic conflicts, *Review of Economic Studies*, Vol. 73, 505-529.

Raith, M., 2003., Competition, risk and managerial incentives, *American Economic Review*, Vol. 93,1425–36.

Spence, M. A., 1981, The learning curve and competition, *Bell Journal of Economics*, Vol. 12, 49–70.

Stefanou, S.E, 1988, Stochastic learning by experience and returns to scale, *Journal of Economic Dynamics and Control*, Vol. 12, 137-143.

Sundaresan, S., and N. Wang, 2007, Investment under uncertainty with strategic debt service, *American Economic Review*, Vol. 97, 256–261.

Thompson, P., 2001, How much did the liberty shipbuilders learn? New evidence for an old case study, *Journal of Political Economy*, Vol. 109, 103–137.

Thronton, R., and P. Thompson, 2001, Learning from experience and learning from others: An exploration of learning and spillovers in wartime shipbuilding, *American Economic Review*, Vol. 91, 1350–1368.

Wakker, P., 2010, *Prospect Theory for Uncertainty*, Cambridge University Press, Cambridge, UK.

Wasserman, L.A. and Kadane J., 1990, Bayes' theorem for Choquet capacities, *Annals of Statistics*, Vol. 18, 1328–1339.

Webbink, D., 1977, *The Semiconductor Industry*, Federal Trade Commission Staff Report, Washington, DC.

Weeds, H., 2002, Strategic delay in a real options model of R&D competition, *Review of Economic Studies*, Vol. 69, 729-47.

Wright, T.P., 1936, Factors affecting the cost of aeroplanes, *Journal of Aeronautical Sciences*, Vol. 3, 122–128.

Zhdanov, A. 2007, Competitive equilibrium with debt, *Journal of Financial and Quantitative Analysis*, Vol. 42, 649–672.

Zimmerman, M. B., 1982, Learning effects and commercialization of new Energy technologies: The care of nuclear power, *RAND Journal of Economics*, Vol. 13, 297–310.